DOCUMENT RESUME

ED 178 313 SE 029 180

AUTHOR	Stelzer, John
TITLE	Structuring and Classifying the Content of Instructional Subject Matter, Volume I. Final Technical Report.
INSTITUTION	Human Resources Research Organization, Alexandria, Va.
SPONS AGENCY	Army Research Inst. for the Behavioral and Social Sciences, Arlington, Va.
REPORT NO	HumRRO-FR-ED-75-17
PUB DATE	27 Jun 75
CONTRACT	DAHC-19-74-C-0066
NOTE	161p.; For related document, see SE 029 181; Contains occasional light type
EDRS PRICE	MF01/PC07 Plus Postage.
DESCRIPTORS	Classification; *Content Analysis; Educational Theories; Instruction; *Instructional Design; Models; Structural Analysis; *Technical Reports

ABSTRACT

A model of instruction is discussed and developed. The model has five requisite components; they are descriptions of: (1) the instruction agent's subject matter representation; (2) the learner's initial and current subject matter representations; (3) the instructional process itself; (4) the state of the learner at any point in the instructional process; and (5) the transition process of the learner as a result of the instructional process. The focus of the document is on component 1 and a theory for structuring subject matter content is developed. This theory is used as a basis for developing a classificatory schema for subject matter content. (MK)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED178313

FR-ED-75-17

"PERMISSION TO REPRODUCE THIS
MATERIAL HAS BEEN GRANTED BY
Barbara Vermillion

TO THE EDUCATIONAL RESOURCES
INFORMATION CENTER (ERIC)."

U S DEPARTMENT OF HEALTH,
EDUCATION & WELFARE
NATIONAL INSTITUTE OF
EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL NATIONAL INSTITUTE OF EDUCATION POSITION OR POLICY

FINAL TECHNICAL REPORT

VOLUME I

STRUCTURING AND CLASSIFYING THE CONTENT OF INSTRUCTIONAL SUBJECT MATTER

Prepared by

John Stelzer

for

US Army Research Institute for the
Behavioral and Social Sciences
1300 Wilson Blvd. (PERI/OU)
Arlington, Virginia 22209

Contract No. DAHC19-74-C-0066

Human Resources Research Organization
Eastern Division
300 North Washington Street
Alexandria, Virginia 22314

June 27, 1975

SE 029 180

CONTENTS

INTRODUCTION

Chapter I: Axiomatically Structuring Subject Matter Content

Chapter II: Classifying Subject Matter Content

Appendix I: The Structure of Axiomatic Theories

Appendix II: Matrices for the Photography Example from Chapter I

Introduction

In the most general terms instruction may be viewed as an interactive process taking place between one or more instructional agents (human or not) and one or more learners (human or not). For simplicity we will refer simply to the instructional agent and the learner, implicitly implying thereby the possibility of plurality in either case. The instructional agent may play no role in the instructional interaction or may contribute to this process. In the former case the learners learn either through their own interaction, as a group, or through individual initiative. In the latter case the instructional agent may play a direct role in the instructional interaction. This role may be to schedule or sequence the interaction; to provide tutorial instruction; to oversee gaming or simulations; to provide information upon request; or, in general, to guide, direct, provide input to, etc. the instructional process.

The learner in an instructional environment, as distinct from the instructional agent, must not be passive. Total passivity on the learner's part, or the removal of all learners from the environment is tantamount to the creation of a non-instructional environment. A learner is necessary in an instructional environment but an instructional agent is not. In other words, instruction cannot take place without a learner to learn, but can proceed without an instructor to instruct.

Two distinct modes of instruction are discernable within this framework. On the one hand the instructional agent may dominate the instructional interaction by simply explaining concepts, ideas, etc. to the learner in a sequence or order selected by the instructional agent. On the other hand, the learner may be left entirely to the learner's own devices to browse through instructional material as the learner sees fit. The former mode of instruction will be referred to as Information Transfer and the latter, Self Discovery.

Between these two extremes lies a continuum of instructional modes. Toward Information Transfer and away from Self Discovery we find what may be called Tutorial/Query instruction. In this mode the instructional agent assumes major responsibility for providing explanations, examples, etc. and for sequencing instruction. The learner may be permitted to request definitions, review or recaptulate material and examples, and, in general, to make some choices in selecting material to be studied.

At the other end of the spectrum, toward Self Discovery and away from Information Transfer, we can define Guided Self Discovery. Guided Self Discovery would result when the learner exerted maximal influence in selecting instructional materials, and in formulating concepts, etc. based on experiences such as gaming, simulation, etc. In such an instructional mode the role of the instructional agent would be minimal, but not nonexistent. The instructional agent would perhaps make suggestions, give hints, provide data on demand and, in general, act as a benevolent, omniscient overseer.

Implicit in this schema is the notion of a subject matter and its structure. What is it that learners aspire to learn? In the most general terms--a subject matter. That is, instruction must be about something and that something we call the instructional subject matter. The subject matter can literally be anything that learners can learn, or even be interested in learning about. "Tell me how telephone poles get where they are and what they are for." "Teach me how to tell time." "Explain Einstein's Theory of Relativity." "What is electricity?" "Explain Transformational Grammar." "Teach me Boolean Algebra." Mommy, where did I come from?" "Eh, what's this about a missing link?" "Behaviorism, doesn't that have something to do with utopia?" "Explain inflation and its relation to recession and oil prices." ... The possibilities are endless.

As a component within an instructional environment the subject matter must be organized coherently. For example, we don't set out to answer the missing link query by saying:

> "Australopithecus. Chimpanzees too. Did you hear about the old bones they found last week in the Rift Valley. By the way don't overlook Olduvai..."

Quite to the contrary, we would expect a competent instructor to first point out the nebulousness of the missing link concept. Subsequent interaction, we would expect, would begin to replace this faulty notion with a coherent picture of humanoid evolutionary trends as espoused by contemporary theorists.

We assume then that the instructional subject matter is organized by or for the instructional agent. To organize a subject matter is to identify its constituents (concepts, notions, etc.) and their interrelationships. This complex of elements and nexuses must be available to the instructional agent. Thus, there must be a representation of the instructional subject matter. This representation may be physical (a book, computerized data, charts, etc.) or non-physical (merely thoughts). The representation may be completed before instruction commences or may be emerging simultaneously with instruction.

We do not require the subject matter representation to be unique. Different instructors bring different understanding to the same subject matters. Thus, it seems natural to suppose that various representations of the same subject matter are to be expected. In fact, the instructional agent may have available more than one representation of the same subject matter. Various representations are one way of beginning to tailor instruction to fit individual learner idiosyncrasies. We do expect the various representations to satisfy certain criteria of internal consistency. One of our goals herein is to specify clearly what we mean by internal consistency of a subject matter representation for instruction.

What happens to the learner as a result of the instructional process? In general, learners develop their own representation or representations of the instructional subject matter. That is, the learner picks out or identifies concepts, notions, etc. in the subject matter, and assimmilates and interrelates these elements. Thus, the learner begins to organize and, hence, to represent the subject matter due to the instructional interaction. Like the instructional agent, an individual learner may develop more than one subject matter representation. Furthermore, we expect that each of these representations will exhibit some form of internal consistency. A learner may begin this development process from a completely naive point of view in which case the initial learner representation would be nonexistent. Alternately, a learner may begin with some preliminary understanding of the subject matter in which case we expect the representation to expand through the addition of new elements and relationships.

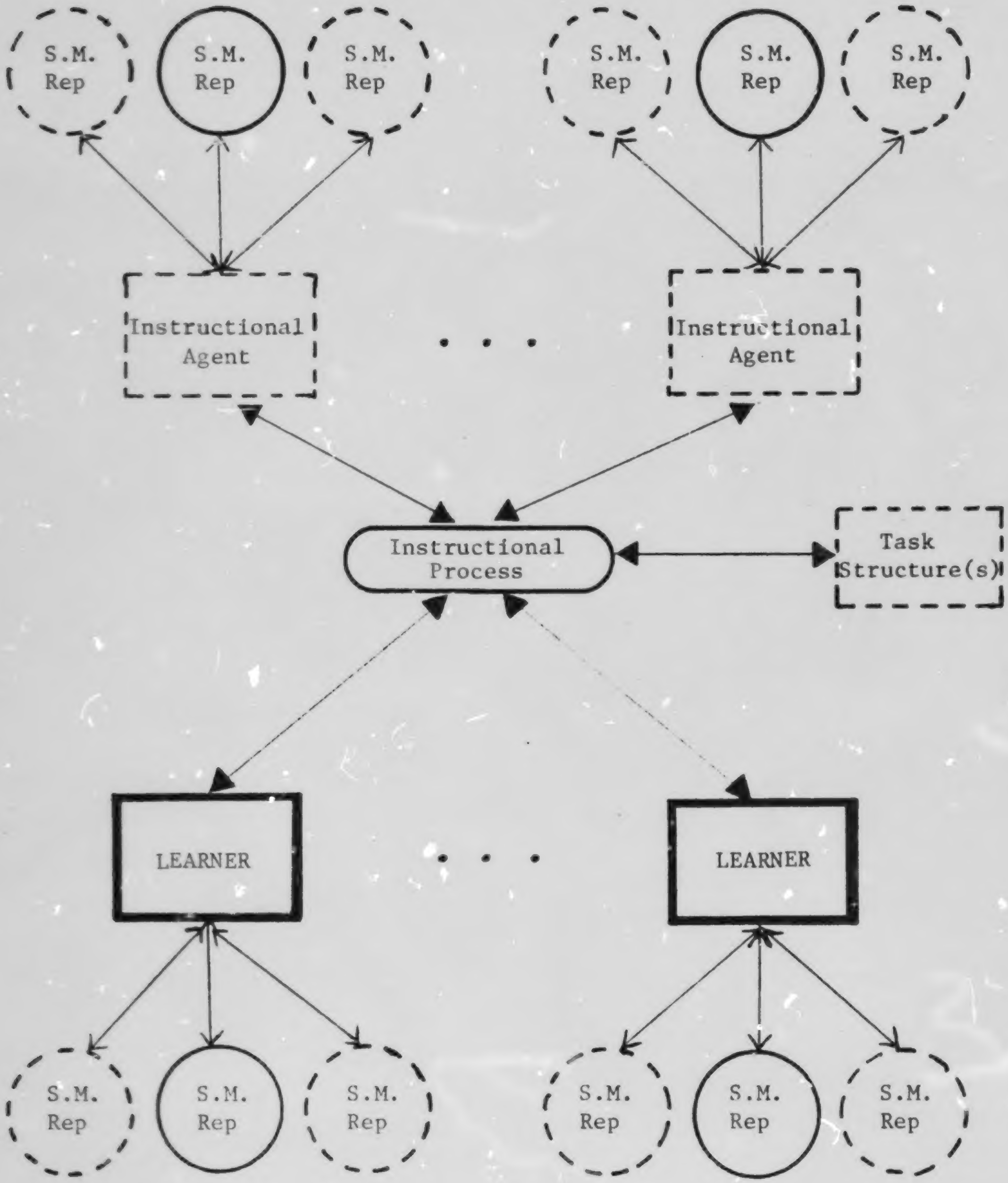
We do not expect a learner's emerging subject matter representation to be compatible with the instructional agent's representation. When we learn we characteristically reorder and even rename notions, concepts, etc. We each bring differing experiential histories, differing conceptualizing processes, and differing knowledge and belief systems to a learning environment. Thus, we expect different learners to learn differently, and to organize what they learn differently. Thus, the learner's task is to construct an internally consistent representation (or representations) of the subject matter from possible input from the instructional agent; this input being based on perhaps incompatible but internally consistent representations of the same subject matter.

We summarize this discussion in Figure 1 where we show the major components that have been singled out in the above discussion. Figure 1 exhibits one or more learners, one of whom must be present--solid boxes. Each learner has at least one subject matter representation (S.M. Rep)--solid circle; and perhaps more than one--broken circles. The instructional agents are not necessary--broken boxes; but there may be more than one. If there are not instructional agents, then the directed lines are to be thought of as traversing the dashed boxes and connecting with at least one subject matter representation--solid circles. Numerous subject matter representations are possible for each instructional agent--broken circles.

Whence, we have sketched a rough overview of instruction. Our goal herein is to augment this sketch by beginning to develop a precise model of instruction. Figure 1 provides a convenient framework within which we can delineate the requisite components for such a model. Our components are very similar to those identified by Pask (1969).

(1) A description of the instructional agent's subject matter representation.

(2) A description of the learner's initial and current subject matter representations.


S.M. Rep
S.M. Rep
S.M. Rep
S.M. Rep
S.M. Rep
S.M. Rep
Instructional Agent
. . .
Instructional Agent
Instructional Process
Task Structure(s)
LEARNER
. . .
LEARNER
S.M. Rep
S.M. Rep
S.M. Rep
S.M. Rep
S.M. Rep
S.M. Rep

OVERVIEW OF INSTRUCTION

Figure 1

(3) A description of the instructional process itself including the goal of instruction and the role of tasks.

(4) A description of the state of the learner--initial and current-- at any point in the instructional process.

(5) A description of the transition process of the learner as a result of the instructional process.

In the overview of instruction sketched above, we have discussed content only, excluding tasks. References to learner states, as above, immediately raise the problem of ascertaining exactly what state a learner is in. Tasks, whose performance is behaviorally observable and, hence, determinable, provide this information. We view task structures as being brought into play during the instructional process to ascertain learner comprehension of instruction units (concepts, notions, etc.). Whence, Figure 1 shows a broken box labeled task structure connected to the instructional process box. While we have not graphically depicted the connection, the task structures also will be closely related to the subject matter representations.

In the chapters to follow, we will begin to develop the above components of the model. In particular, we will focus on component 1 developing a theory for structuring subject matter content. On the basis of this theory, we will develop also a classificatory schema for subject for subject matter content. This schema in part will be based on the notion of an analogy for a subject matter. This notion is investigated in detail herein.

The model components developed herein are descriptive in nature. We will not provide only descriptive analyses, but we will also provide prescriptions. Prescriptions for the instructional agent for constructing a subject matter representation are also provided.

Chapter I

AXIOMATICALLY STRUCTURING SUBJECT MATTER CONTENT

As discussed in the Introduction the instructional process can be viewed as an interaction between a human or non-human instructional agent and a learner, also, either human or not. Effective instruction results when the learner learns what the instructional agent intends the learner to learn. Effective instruction, it seems, is neither random nor whimsical. Indeed, when the instructional process is effective it seems that precise, methodological principles are followed. While we may agree with this assertion, the problem of describing the principles that lead to effective instruction has proved to be highly intractable.

Methodological principles of this kind cannot be formulated in a vacuum. A coherent model for the instructional process is required in which principles may be formulated, discussed, tested, and either verified or falsified. This means that a descriptive model for instruction is required. Given such a descriptive model it is possible to experiment with methodological principles drawn from the model. As a result it would be expected that prescriptions for conducting the instructional process would be forthcoming from the descriptive model.

We have identified in the Introduction four components we would expect to be included in a descriptive model of instruction. In this Chapter we will be concerned with the first of those components: describing the instructional agent's representation of subject matter content.

As asserted above, instruction is an interaction between an instructional agent or instructor, and a learner. Instruction is always about or concerned with some particular subject matter. The instructional agent must in some sense have available at least one representation of the subject matter. That is, the subject matter must be broken down into a set of concepts, ideas, notions, relations, etc., which are available to the instructional agent. These constituents, hopefully, are then in some sense transferred to the learner as a result of the instructional process. Thus, the instructional agent must have a representation of the subject matter; and must have some structure embedded in this representation so that transferral to the learner can be effected systematically. Without an internal structure to follow, the instructional process can be expected to bog down in a wealth of confusion. Put another way: It does not seem sensible to expect instruction in complex subject matters to be effective, if the subject matter concepts are taught in haphazard order.

This does not imply that every subject matter has a unique representation, or even a unique underlying structure. On the contrary, an instructor is free to structure a particular subject matter as the instructor deems best. It can be expected that different instructors will comprehend differently the same subject matter. Since the instructors have differing comprehension, we expect their representations and structurings of the subject matter also to differ. As different as these structurings may be, they may all be used to provide equally effective instruction in the same subject matter to different learners, or even to the same learners. What we are asserting is that given the fact that each instructor has a structure for the subject matter, then it follows that that structure restricts the instructional process in precise, identifiable ways.

Our model does not imply that the instructional agent must also have the complete subject-matter representation available at the outset of the instructional process. The model admits the possibility that the instructional agent is "just keeping ahead" as it often is referred to. Not that we condone this practice, but since it is so widespread in the "real world", the model under development might be rejected if this possibility were eliminated. Even in this deplorable case, we assert that the instructional agent's developing representation of the subject-matter will display coherent organization and structure.

Now consider the learner. The learner also has a representation or description of the instructional subject matter. As with the instructional agent, this representation will be coherent and structured. Initially, we might expect the learner to be completely naive in that the learner possesses none of the concepts for the subject matter and, hence, no structure. On the other hand, the learner may have some of the notions and some structure in which case we would say that the learner is not totally naive. As a result of the instructional process, we expect the learner's representation of the subject matter to be altered in some way. The number of subject-matter concepts familiar to the learner may be increased. Also, the structure imposed by the learner on the subject matter may be enriched or augmented. In the latter case this enrichment may take place without concomitant growth in the number of comprehended subject-matter constituents. This would occur if new interrelationships within the subject matter were made evident to the learner.

Thus, we see that the instructional agent will possess a representation, with structure, of the subject matter. This representation may be static or growing. The learner also has a representation with structure of the subject matter. We expect this representation to undergo change as the result of the instructional process. In fact, failure to alter the learner's representation implies, it seems, ineffectiveness of the instructional process.

Diagramatically, we have represented this model overview in Figure 1 of the Introduction. The solid boxes in the figure are intended to denote real-world entities that must be present. Dashed figures represent elements that may or may not be present. That is, we assume the instructional agent and the learner to be existent entities. The instructional process is then assumed to be a process that occurs in the real world. We intend to avoid falling into a long debate concerning mental constructs which the subject-matter representations may well be, on either the instructor's or learner's side. For us, it does not matter if a subject-matter representation is a mental construct, a bit pattern in a computer memory, or a pencil-and-paper diagram. What is important for us is that no matter what the medium of representation, the representation will satisfy certain principles, and our problem is to discover and describe those principles.

We hasten to point out that we will not even require the instructor's and learner's subject-matter representations to be compatible! In general instruction does not simply consist of a transferral of the instructor's subject-matter representation to the learner. On the contrary, we view instruction as a process in which the instructor displays a subject-matter representation in a way that allows the learner to develop the learner's own subject-matter representation. The structure imposed on the subject matter by the learner may be completely different from that imposed by the instructor. These two different structures may in fact be inconsistent with one another. But we do expect each structure to exhibit internal consistency. Again, our problem is to discover and describe the principles that underlie subject-matter representation and structuring, and ensure that the structure is internally consistent.

Principles must also be developed that will serve to guide and direct the instructional process. That is, one aspect of the instructional model being developed is to provide rules by which the instructional process must proceed in order to avoid having the instructional process degenerate into a meaningless interaction. We may not be able to guarantee transferral, but we should be able to specify how to avoid mass confusion.

The developments in Appendix I provide a theory through which we can begin to develop these model components. Axiomatic methodology provides a framework within which we can discuss the principles concerned with representing and structuring subject matter. These developments in turn provide a framework in which we can formulate principles to control the instructional process. More precisely, we have an abstract, formal theory for structuring subject matter. This abstract theory can be interpreted to provide a theory for representing and structuring instructional subject matter. The theory also provides us a means for describing internal consistency as it relates to structured instructional subject matters. This theory will also lead to other models that can be used for controlling the instructional process. We expect these various axiomatic theories to provide an initial descriptive model for instruction; a model that satisfies each of our five components. and component (1), in particular.

We restrict our attention in this chapter to the problem of representing and structuring instructional subject matter for the instructional agent. The developments reported herein are an outgrowth of earlier work which we now describe and summarize. Our theoretical developments follow this summary. We conclude this chapter with a formulation of prescriptions for structuring subject matter for instructional purposes. These prescriptions can be seen to be consequences of the descriptive model we develop.

Stelzer and Kingsley (1974a,b) have developed a theory of subject-matter structure based on the paradigm of axiomatics. The concept underlying this theory is that axiomatics has been employed for some 2,500 years by scientists to structure subject matters such as geometry, physics, economics, kinship relations, psychological measurement scales, probability, etc. Thus, to apply axiomatics is to structure a subject matter. The converse does not necessarily obtain since subject matters may be structured in ways other than axiomatically. However, Stelzer and Kingsley argue that axiomatics does provide a paradigm which can be applied profitably to the problem of structuring subject matter content for instruction.

Stelzer and Kingsley are primarily concerned with the problem of structuring subject matter as it pertains to instructional applications. They seek a method an author may follow that will result in structured instructional material. The method they develop is based on the way in which axiomatics proceeds. Appendix I can be seen to be an extension of this earlier work. Stelzer and Kingsley do not analyze the axiomatic process as completely as has been done in Appendix I. Stelzer and Kingsley have developed and described, however, essentially the same matrix approach for enumerating dependency relationships that is described in Appendix I.

The results reported herein will diverge from the original Stelzer/Kingsley developments in three ways. First, in the description of the axiomatic paradigm, Stelzer and Kingsley do not distinguish between symbols and statements. For example, they do not differentiate a defined symbol from its definition. They treat symbols as statements containing one expression. This approach was taken in order to simplify their theoretical developments. Subsequent investigations have lead to the belief that this simplifying assumption may not be warranted since relevant instructional information may not be forthcoming. Thus, it was felt that a deeper, more sophisticated analysis of axiomatics was required.

Appendix I presents the results of this further analysis of axiomatics, dropping the original simplifying assumption. In this new theory the expressions and statements within an axiomatic structure are clearly delineated. The expressions in an axiomatic theory are seen to represent either primitive or defined concepts, while statements are axioms, definitions (of defined concepts), or theorems. It can be seen that this analysis proceeds on the basis of eight primitive concepts and twelve axioms. A theory for a subject matter that contains these concepts and that satisfies the axioms is referred to as an Axiomatically Structured Theory. We will show presently how the concept of an Axiomatically Structured Theory can be interpreted to provide a theory for structuring subject matter content for instructional purposes.

The second manner in which the present theory diverges from that of Stelzer and Kingsley concerns the role that tasks play in the subject matter structure. As Stelzer and Kingsley point out there seem to be two categories of earlier attempts to characterize subject-matter structures. Theories such as Gagne's have focused on task structures. Gagne's formula, "what must one be able to do before...?" leads to task oriented structures. Pask working from the formula, "What must a person know before learning about...?" leads to content oriented structures.. Stelzer and Kingsley analyze subject matter structures into both content structures (their GCN) and task structures (their TICS). Coordinating relations serve to interrelate these model components within their theory. The complete complex of GCN, TICS and Coordinating Relations is for Stelzer and Kingsley the subject matter structure.

The theory we will develop does not include the task structure as part of the subject-matter structure. In our theory, we will include in the representation of the subject matter, subject-matter content components only. That is, our representation theory will be an improvement over that portion of the Stelzer/Kingsley theory that deals only with the GCN. From the point of view of a more comprehensive theory of instruction, along the lines of Figure 1, this seems to be a preferable approach. The learner's representation of the subject matter simply would not be expected to include a task structure. Rather than impose an unnatural asymmetry on Figure 1, we will not require the instructor's subject-matter representation to include a task structure either.

We intend to relocate the task structures in the instructional model we are developing. While these developments are beyond the scope of this chapter, we will indicate briefly how these structures will be included in the theory. The instructor will have a representation of the content of the subject matter as shown in Figure 1. The task structure will be available to the instructional agent during the instructional process. As in the original theory, coordinating relations will relate the instructional agent's subject-matter representation with the now relocated task structure. Tasks will play the role of ascertaining content constituent comprehension on the part of the learner via behavioral observations. The structure exhibited by the tasks will be closely allied with the structure exhibited by associated content elements. The details concerning these modifications must await future model developments. We now return to the mainstream of our developments.

The third divergence in the present theory from Stelzer and Kingsley's initial formulation is a subtle shift in emphasis. It should be noted that

Stelzer and Kingsley do not provide a theory to be used *when* developing instructional material. Their point of view is that given existing material they can extract from that material the components that fit their descriptive model. This in turn leads to the dependency hierarchy. The point of view developed herein is that if one desires to structure subject matter, then the appropriate model for this process is the axiomatic method. Thus, in developing subject matter one should play a role akin to axiomatizing a subject matter. This position is decidedly stronger than that expressed by Stelzer and Kingsley. We assert that the axiomatic paradigm is the appropriate paradigm to follow when confronted with the problem of structuring subject matter for instructional purposes, especially for Information Transfer instruction.

Whence, we have arrived at our point of departure for our theoretical developments. How is it that axiomatic methodology provides the appropriate paradigm for structuring subject matter? Since Appendix I provides an analysis of the structure of axiomatic theories, our task is reduced to the problem of showing how to apply this model to the problem of structuring subject matter. We will show how to structure instructional subject matter along the lines of the axiomatic paradigm in such a way that the resulting subject-matter structure becomes a *model* for the concept of an Axiomatically Structured Theory (Appendix I). Since the instructional subject-matter structure will be a model for this theory, it follows that the results derived in Appendix I will carry over to the instructional subject-matter structure. Thus, the dependency relationship also defined in Appendix I will be forthcoming within the context of instructionally structured subject matters, as will the matrix method for identifying dependency relationships.

Thus, our first goal is to define the concept of an Axiomatically

Structured Subject Matter in such a way that this concept is a model for the concept of an Axiomatically Structured Theory. We have eight primitive concepts to define. To begin with, the concepts, notions, etc., in a subject matter will be formulated within the context of some underlying language, L. In general, the language L will be a natural language such as English or Russian. As a natural language, L will include a lexicon and a grammar that, in turn, includes transformation rules for generating all well formed terminal strings for L.

The language L is intended to be the linguistic vehicle through which the instructional agent and learner communicate. Statements within L are formulated and communicated to the learner. The learner in turn formulates statements within L and communicates these to the instructor. Thus, we expect L to be the natural language for both the instructor and the learner.

It is expected that the learner is to some degree naive about the subject matter. This means that the language for the learner does not contain terms that make references to subject-matter-related concepts. That is, we assume that the learner's natural language does not contain subject-matter-related terminology. This would be the most general case in which the learner is totally naive concerning the subject matter. From a conceptual point of view, not having subject-matter-related terminology available means that the learner is unacquainted with subject-matter-related concepts.

The instructor on the other hand is familiar with the subject-matter-related concepts. Accordingly, the instructor has available subject-matter-related terminology that is used to represent the subject matter concepts. Thus, the instructor, as distinct from the learner, has augmented L to include these references. The instructional problem, in part, can be viewed in one way as being a problem concerning the augmentation of the learner's language so that it also includes subject-matter-related references. We desire to extend the learner's conceptual understanding so that it encompasses the subject matter. Thus, we must concern ourselves with how L is to be augmented to permit subject matter references.

For a given subject matter, some notions concerning the subject matter will be assumed to be intuitively understandable in that they will be utilized without being defined. These are referred to as the Primary Notions for the subject matter. They represent the conceptual basis on which rests the subject-matter structure that will develop. Each Primary Notion will be represented by a unique Primary Term. The Primary Terms interact syntactically with other terms in the subject matter to produce statements, etc., as will be described below. The set of Primary Terms is represented by PT.

The Primary Notions for a subject matter can be seen to provide a commonality between the instructor's and learner's conceptual frameworks. We may not be able to say with certitude exactly what each Primary Notion means. But the instructor at least must be willing to assume that whatever the instructor means by a Primary Notion it will mean about the same for the learner. These Primary Notions serve as a conceputal starting point from which instruction may proceed.

It may well be that an instructor has a Primary Notion that is not Primary for another instructor. In other words, we see no compelling reason to believe that a unique set of Primary Notions will be forthcoming for a particular subject matter. This point has an analogue in axiomatics where there is complete freedom in the choice of a set of primitive concepts for axiomatizing a theory. Thus, one instructor's subject-matter structure may differ from another instructor's for the same subject matter in that they proceed from different sets of Primary Notions.

Besides Primary Notions, we expect that other notions concerning the subject matter will be introduced. These notions are Secondary, or Defined, in that their meaning is dependent on the meaning assigned to other notions, either Primary Notions or other Secondary Notions. Each Secondary Notion is represented by a unique Secondary Term, with the set of Secondary Terms being represented by ST. As with Primary Notions, each instructor is free to select and introduce those Secondary Notions of the instructor's choosing. Thus, we do not expect two instructors to introduce the same Secondary Notions for the same subject matter.

In selecting Primary and Secondary Terms to represent Primary and Secondary Notions, an instructor is faced with two possibilities. On the one hand a Term that already appears in the lexicon of L may be chosen to represent a Notion. The use of such a term will be governed by transformation rules in the lexicon for L. If the new use of the old Term functions syntactically as in the old usage, then the transformation rules for L can remain unchanged. But if the new use of the old Term functions syntactically in new ways, then the transformation rules for L need to be altered to reflect these altered syntactic possibilities. Of course, on the other hand, if an entirely new Term is introduced to represent a Notion, then transformation rules governing its use will be required.

Thus, we see that PT and ST serve to extend L to a new language L^1. The language L^1 may have as a lexicon an extension of the lexicon for L, in which case transformation rules are added to the grammar of L to produce the grammar for L^1. Some Terms in either PT or ST may already appear in the lexicon of L in which case it may be necessary to alter the transformation rules for L to permit the appropriate syntactic use of Primary or Secondary Terms. In any case, it is important for an instructor to be aware that in structuring subject matter a new language is being developed out of the underlying language available to both the instructor and learner.

Within L^1 it will be possible to formulate some set of statements. This set of Possible Statements, PS, consists of those Statements that are well formed syntactically according to the grammar for L^1. In general, PS will be an infinite set. In structuring a subject matter, an instructor actually will formulate some set of Statements, S, which is a subset of PS. Since only a finite number of Statements can be formulated in a finite length of time, S will be finite. We will say that elements of S express or represent facts concerning the subject matter.

At this point we have identified four sets which along with the language L represent our first five primitive concepts. These elements are as follows.

L - the underlying language

PS - the set of Possible Statements in L^1 or L extended

S - the set of Statements actually formulated by the instructor in L^1.

PT - the set of Primary Terms representing Primitive Concepts for the subject matter

ST - the set of Secondary Terms representing Defined Concepts for the subject matter.

Our last three primitive concepts are relationships that serve to relate these entities. These relationships are:

"used to formulate"

"used to prove"

"can be used to prove."

It may be that Primary or Secondary Terms are used in the formulation of a Statement. In this case we say that the Term "is used to formulate" the Statement. That is, if t is a Primary or Secondary Term that appears in Statement s, then we say that "t is used to formulate s" and we write Fts. Thus, we see that this relationship is purely syntactical. In the phrase Fts, t always refers to either a Primary or Secondary Term and s always refers to either a Statement or a Possible Statement.

In general, Statements concerning a subject matter introduced by an instructor will be one of three types. A Statement may be assumed to express a valid characteristic of the subject matter, in which case it is analogous to an axiom in an axiomatic theory. A Statement may be introduced as a definition of a Secondary Notion, in which case it is analogous to a definition in an axiomatic theory. Finally, a Statement may express a valid characteristic of the subject matter, but may do so by virtue of the fact that it is proved. In the latter case the introduced Statement, s, will be proved from a set, A, of already available Statements. We will write "$A \vdash s$" to mean Statement s is proved on the basis, or from, the set of Statements A.

An instructor, when structuring a subject matter, must also be cognizant of the possibility of proving certain Statements from sets of Statements. We will write $A \models s$ to mean that Statement s can be proved from the set of Statements A. In actual practice this relationship may be of little importance but it is of significant importance for the developments to follow.

To summarize, the three relationships interrelate elements as follows.

F relates Primary or Secondary Terms ($PT \cup ST$) to
Possible Statements (PS).

$\vdash$ relates sets of Statements (subsets of S) to individual
Statements (S).

$\models$ relates sets of Possible Statements (subsets of PS) to
individual Possible Statements (PS).

As can be seen these eight concepts correspond to, and have the same structure as, the eight concepts providing the basis for the concept of an Axiomatically Structured Theory (Appendix I). We will now introduce the assumptions that we impose on these eight concepts. There will be a total of 12 assumptions.

Assumption 1. PS is the set of all well formed Statements within some Language, L^1, which is a modification of the underlying language L, effected as follows.

(1) Terms appearing in PT or ST that do not appear in the lexicon of L are added to the lexicon of L; and transformational rules appropriate for these Terms are added to the grammar for L.

(2) Transformation rules in the grammar for L for Terms appearing in either PT or ST, and in the lexicon of L, are modified as required to permit new transformational possibilities.

(3) Step (2) is carried out in such a way that non-subject matter related uses of these terms remain unchanged so that every Statement expressible in L remains expressible in L^1.

(4) Primary and Secondary Terms as members of the lexicon of L^1 are not in and of themselves Possible Statements.

The meaning of assumption 1 should be clear from the foregoing discussion. The next assumption asserts the finiteness of the set of Statements, and that the set of Statements is a subset of the set of Possible Statements. This assumption is reasonable since we do not want the set of Statements to include any illegitimately formed Statements. Also, we have already remarked that S must be finite since finitely many Statements can be formulated in a finite period of time. We also assume that the Statements, S, are distinctly labelled so that their labels can be used to identify them. Thus we have the following assumption.

Assumption 2. The set of Statements, S, is a finite subset of the set of Possible Statements, PS. The elements of S are distinctly labelled so that different elements have different labels.

The next two assumptions are concerned with PT and ST. PT will be finite since the number of intuitively understandable Notions on which subject matter is based surely cannot be infinite. PT must also be non-empty or else there will be no conceptual basis for the subject matter. Finally, PT and ST must be mutually exclusive since it would be an error to permit the same Term to represent both an undefined, Primary Notion and a defined Secondary Notion. Whence we have the following two assumptions.

Assumption 3. The set of Primary Terms, PT, is both finite and non-empty.

Assumption 4. The same Term is not used to represent both a Primitive and and Secondary Concept. That is, PT and ST have no common members.

As asserted in assumption 4 the set of Primary Terms must be disjoint from the set of Secondary Terms. Primitive Concepts within a subject matter play a role that is clearly distinct from the function played by Secondary Concepts. Primitive Concepts, represented by PT, are intended to be intuitively comprehensible. Defined Concepts, represented by ST, are syntactical abbreviations to simplify the exposition of the subject matter.

Elements in ST, as to be discussed in detail, are entirely expendable since they always can be replaced with expressions built up from PT only.

It is expected that every Defined Term will be used in some way within the subject matter. Otherwise, the subject matter is unnecessarily obfuscated through the inclusion of unemployed defined conceptualization. Defined Concepts require a Definition that introduces their associated Defined Terms. Thus, in general, every Secondary Term will be used to formulate at least one Statement, its Definition. This is the rationale that underlies the next, and fifth, assumption. We will assume that each Secondary Term is associated with a unique Statement, to be defined below as the Term's Definition. This association is unique in that given a Term we can identify its Definition; and, conversely, given a Definition we can identify the Secondary Term it defines.

Assumption 5. There is a correspondence between ST and S such that each element in ST is associated with a unique element in S.

As will be seen assumption 5 eliminates the possibility of Defining more than one Secondary Term in one Statement. We need to be able to refer to the Definition for each Secondary Term. Such a Definition will be a Statement within L^1. If we allow more than one Secondary Term to be Defined in one Statement, then we require some method of separating and distinguishing the Definitions. Such a method does not seem to be available. For example, it does not seem possible to separate a multiple Definition by syntactic devices such as separating sentences. Some Definitions take several sentences to state. The conglomerate of sentences would be labelled as one Statement and would count as the Definition. Rather than become embroiled in such a seemingly hopeless pursuit we will base our developments on the correspondence postulated in assumption 5.

It is recognized that in practice this approach is often violated; but in an insignificant way. A subject matter representation may have a labelled definition in which two or more Concepts are defined. In such cases, however, it is readily seen that the Definition can be split into two distinct definitions, preserving the validity of our approach. In general, it will be true that Definitions introducing multiple concepts can be replaced by a series of Definitions with each Definition introducing only one Secondary Term. Thus, while artistic preference may lead to violation of our approach, we contend that these violations are trivial in nature, and that, in fact, it is reasonable to base the concept of a definition on assumption 5. In other words, we might say "the letter, but not the spirit of the approach we take is sometimes violated in practice".

Definition 1. The Statement corresponding to a Secondary Term, t, postulated in assumption 5 is referred to as the Definition of t and is denoted by t*.

It follows from assumption 5 that distinct Secondary Terms have distinct Definitions. Thus for all t_1, t_2 in ST, $t_1 \neq t_2$ if and only if $t_1^* \neq t_2^*$. From assumption 2 we know that S is finite. From this it follows that ST must be finite. Otherwise, due to the uniqueness of the correspondence associating ST and S, S would be infinite. We state this as a theorem.

Theorem 1. The set of Secondary Terms, ST, is finite.

Since t* is the definition of Secondary Term t for all t in ST, we expect t to appear in t*. That is, we expect t to be used to formulate its own Definition. This is the substance of assumption 6.

Assumption 6. Every Secondary Term is used to formulate its own Definition.

Assumption 6 can be restated as follows. For all t in ST, Ftt^* holds.

Notice also that these notions concerning Definitions are restricted to elements of ST only, excluding elements of PT. Even when a Primitive Term is used to formulate several Statements none of these elements are to be construed as the Definition of the Term. A Primitive Concept may be used in several axioms and it is incorrect to select any one of these as the Definition of the Concept. The entire collection of axioms in a sense characterize the Primitive Concepts and their interrelationships; but they do *not* Define these notions. The Primitive Terms in a subject matter represent the undefined, intuitively comprehensible conceptual basis for the subject matter.

It may be felt that assumptions like the fifth and sixth assumptions are required for Primary Terms as well. This is not the case. Primary Terms represent intuitively understandable Notions, and, hence, do not require Definitions. Also an instructor is free to single out a Primary Term to represent a Primary Notion and never use the Term in a Statement. Strange as it would be to introduce unused conceptual baggage into an instructional setting, there is *theoretically* nothing wrong with doing so. The practice should be avoided, of course.

Our next assumption concerning Definitions is simply that they are unproved Statements. Following Suppes, we perceive Definitions as functioning like axioms within an axiomatic theory. Definitions serve as additional suppositions for subsequent developments. Whence, it is entirely reasonable to assume that Definitions are not proved.

Assumption 7. Definitions are not proved.

Assumption 7 can be restated by saying that for all t in ST, it is not the case that there is a subset, A, of S such that $A \vdash t^*$ holds.

We next direct our attention to the questions of the eliminability of Defined Concepts and the non-creativity of Definitions. The concepts underlying these notions are quite simple. Consider the question of the elimination of Defined Concepts. As mentioned earlier Defined Concepts are intended to serve as syntactic abbreviations for other, usually more complex, syntactic formulations. For example, in photography the notion of a "camera" may be defined in terms of the notions of a "lens", "shutter", "film", "device", and "contained within":

> A camera is a device containing a lens, a shutter and film.

In this Statement "camera" is the Defined Secondary Term representing the concept of a camera. To say that "camera" is eliminable means, first, that every occurrence of "camera" in every Possible Statement within the subject matter can be replaced with a syntactic construction not containing "camera". The above "definition" of "camera" is an equivalence and, thus, every Statement containing "camera" can be rewritten without "camera" simply by substituting the expression "a device containing a lens, shutter and film" for "camera". In practice Definitions usually take the form of logical equivalences or logical identities rendering this substitutions process straightforward.

Suppes (1957) provides a formulation of a "Criterion of Eliminability". Suppes' criterion makes use of the phrase: "preceeding definitions of the theory". Suppes' perceives the Statements in a subject matter as being ordered by this notion of precedence which makes his criteria sensible as long as such an ordering is available. We, however, have not ordered the Statements in any way. That is, the only structure we have available results from the notions of Syntactic and Deductive Dependence to be defined below. These notions are not suitable for the construction of a precedence ordering of the type Suppes requires.

The divergence between Suppes' and our approach can be explained in another way as well. Suppes seems to have based his criteria on some notion of the order in which Statements are written down--say in a book--or communicated. Of course this order may not be the temporal order in which the Statements are formulated. The order is simply the order in which they may be encountered by someone learning the subject matter. We have chosen, in contrast, to treat the questions of inherent structure and communication separately. In this chapter we are treating structure which is not dependent on presentation order. We expect, on the other hand, the order of communication, or presentation, of Statements to depend on the structure exhibited by the subject matter.

The second component of eliminability is that when a Secondary Term, t, is eliminated from a Statement, s, resulting in a Statement, s_1, s and s_1 must be provably equivalent. The equivalence proof must follow from the Definition of the eliminated Term only. That is, it must be possible to prove the equivalence of s and s_1 from t*. We do not expect that such a proof has actually been carried out ($\vdash$) for all Secondary Terms and all Possible Statements. We only require that it can be carried out ($\models$). Thus, this component of the axiom of eliminability must be written as follows.

$$\{t^*\} \models s \leftrightarrow s_1.$$

The analog to this assumption will be found in Suppes' criteria.

Notice also that s_1 need only differ from s in that t in s has been replaced with the terms in t* that Defined t. This means that a Term, x, occurs in s_1 just in case x is not t, and either x is in s or x is in t*. Whence, the syntactic component of the axiom of eliminability is rendered as

Fxs_1 if and only if x is not t and either Fxs or Fxt*.

We have explained the major features of assumption 9 below--eliminability. Before formally stating assumption 9, we introduce another assumption that simply relates the "is proved" and "can be proved" relationships. While simple and straightforward this assumption is required for subsequent developments.

Assumption 8. If a Statement is proved, then it can be proved.

Assumption 8 can be restated as follows.

For all s in S if there is a subset, A, of S such that $A \vdash s$, then $A \models s$.

Assumption 9. (Eliminability) The Definition of any Secondary Term, t, is formulated in such a way that any Possible Statement, s, formulated in terms of t is provably equivalent to another Possible Statement, s_1, that is formulated only with Terms other than t, and that appear either in s or in t*; the equivalence proof using only t*.

Using set theoretic and logical notation we can restate assumption 9 as follows. For all t in ST and for all s in PS if Fts, then there is an s_1 in PS such that for all x in $PT \cup ST$ the following conditions hold:

$$Fxs_1 \leftrightarrow x \neq t \,\&\, (Fxs \vee Fxt^*)$$

and $$\{t^*\} \models s \leftrightarrow s_1 \,.$$

In these phrases "$\leftrightarrow$", "&" and "v" stand for "if and only if", "and", and "or", respectively. The notation "{t*}" means the set consisting of the Sentence t* only.

Now consider the notion of non-creativity. Our formulation of non-creativity will be very similar to Suppes' formulation. However, as with eliminability, Suppes employs the precedence concept, unavailable to us, in non-creativity. The idea behind non-creativity is that Definitions should not enhance substantially the deductive power of the subject matter. We must not be able to prove Statements concerning the underlying subject matter with a Definition that cannot be proved without the Definition. That is, Definitions, being essentially syntactic abbreviations, must not result in new knowledge concerning the subject matter that could not have been proved without the Definitions.

To make these notions more precise consider a Secondary Term, t, and a Possible Statement, s, such that t is not used to formulate s. We might say that s expresses some fact about the subject matter that is entirely expressible without t. Now suppose also that we have a subset of the set of Statements, A, such that t is not used to formulate any Statement in A, and A and t* can be used to prove s. That is, we are assuming that

$$A \cup \{t^*\} \models s$$

holds where "$A \cup \{t^*\}$ is the set consisting of elements from either A or {t*}. To say that t* is non-creative is simply to assert that given these assumptions it follows that

$$A \models s$$

holds. That is, Statements not containing t provable from t and other Statements not containing t, remain provable from the set of Statements without t*. Whence, t* does not deductively strengthen the theory by leading to new results not provable without t*.

Formally, we have the following assumption.

Assumption 10. (Non-creativity) The Definition of any Secondary Term, t, is formulated in such a way that any Possible Statement, s, not formulated in terms of t and provable from t* and a set of Statements, A, no members of which are formulated in terms of t, remains provable from A only.

In other words, for all t in ST and for all s in PS if $\neg Fts$, then for all subsets, A, of PS if

$$A \cup \{t^*\} \models s$$

holds where for all s_1 in A, $\neg Fts_1$ also holds, then

$$A \models s.$$

It would be somewhat more accurate to say that assumptions 9 and 10 assert that Definitional Statements must satisfy weak eliminability and weak non-creativity. It is not entirely evident from these axioms that every Secondary Term is actually eliminable from a subject matter. Consider, for example, a Definitional Statement, t*, and an arbitrary set of proved Statements (theorems), Definitions and other unproved Statements (axioms) such that some of these elements are formulated in terms of t and some are not. It also may be the case that t* is used in some of these proofs which menas that t* is used to derive some Statements. If t is to be truly eliminable, then it must be shown that every single occurrence of t and t* in this group is dispensible. In this sense of strong eliminability, not only must it be shown that equivalent Definitions, theorems and axioms are forthcoming that are not formulated in terms of t; but it also must be shown that the equivalent theorems can still be proved without t*.

It is important to notice that given a subject matter that has been structured, if a Secondary Term t is eliminated from the structure in the above sense, then, in effect, a new structure for the subject matter has been created. That is, suppose that we begin with a representation for a subject matter. Suppose that this representation makes use of a Secondary Term, t, representing a Defined Concept. Further suppose that we eliminate t by dropping the Definition of t, and by replacing every other Statement in which t occurs with an equivalent Statement not using t. In each case assumption 9 assures us that such an equivalent Statement is forthcoming. As a result of this process we have generated an entirely different representation for the subject matter. Conceptually, it is different since the new representation has one fewer Defined Concepts. That is, not only is it different but it is also conceputally simpler. The new representation will also be different in that it will have a different set of Statements, assuming t is originally used in a Statement other than its own Definition.

At this point in our development we can relate the above concepts to the notions introduced in Appendix I. An *Axiomatically Structured Representation* of a subject matter is a representation which exhibits each of the above eight concepts (L, PS, S, PT, ST, F, $\vdash$, $\models$) and which satisfies Assumptions 1 thru 10 above; and Assumptions 11 and 12 to be introduced below. We can show that an Axiomatically Structured Representation for a subject matter satisfies the requirements for the concept of an Axiomatically Structured Theory. That is, an Axiomatically Structured Representation for a subject matter is an Axiomatically Structured Theory. Therefore, *every theorem and result established for Axiomatically Structured Theories (Appendix I) will carry over, and be true of, Axiomatically Structured Representations for subject matters*. Thus, the notions of simplification, reduction, dependence, etc., will be applicable to Axiomatically Structured Representations, as will be the results and methods concerning the identification of dependency relations within an axiomatic theory. Before turning to these details, let us consider how these notions fit into the context of Figure 1, our sketch of an instructional model.

As shown in Figure 1, the instructional agent must have available a representation of the instructional subject matter. The appropriate form for such a representation is to require that it be an Axiomatically Structured Representation for the subject matter. By requiring the representation to be Axiomatically Structured, the instructional agent will have available a precise, unambiguous, conceptual representation for the subject matter; a representation that can provide a basis for systematic instruction in the subject matter. On the other hand, the goal of instruction can be seen to be, in the most general terms, to equip the Learner with an Axiomatically Structured Representation of the same subject matter. The instructor's and learner's axiomatic representations need not be the same, nor even be compatible. Each will exhibit, however, the same internal consistency exhibited by Axiomatically Structured Theories, in general, and by Axiomatically Structured subject matter Representations, in particular.

We will now consider in more detail the internal structure exhibited by an arbitrary Axiomatically Structured Representation, R, for a subject matter. The concepts and results to be discussed correspond to notions and theorems developed in Appendix I. We stress that throughout this discussion R is assumed to be an Axiomatically Structured Representation. Thus, R is associated with L, PS, S, PT, ST, F, $\vdash$ and $\models$, as described above, and R satisfies assumptions 1 thru 10 above. We also assume R satisfies the last two assumptions to be introduced below.

The set of Statements, S, can be seen to be composed of three mutually exclusive sets. One of these subsets of S is the set of Definitions of Secondary Terms. Another of these sets is the set of Results that are actually proved. These Results express facts about the subject matter. Results are distinct from Definitions in that the former are proved, while the latter are assumed to be valid statements, and are not proved. But Definitions are non-creative and by themselves do not lead to interesting results. Interesting Results are forthcoming on the basis of Suppositions concerning the subject matter. Suppositions are also assumed to be true but unlike Definitions are taken as expressing intuitively acceptable facts concerning the subject matter. As such Suppositions are entirely like axioms in an axiomatic theory.

It is possible for each of these sets to be empty. In fact S may even be empty. If S is empty we have the degenerate representation for the subject matter in which nothing is asserted about the subject matter. In this case there is nothing to teach. So, in general, S will be non-empty. If the set of Suppositions is empty, interesting results concerning the subject matter will not be provable. While this is a possibility, such a representation of a subject matter would be of little interest. Similarly, if we have Suppositions but no Results, then the subject-matter Representation is of little interest. In this case, assumptions would be made but they would not be put to use. Finally, we may have Representations in which we have Suppositions and Results, but no Definitions. Such a Representation would be primitively based in that it would employ only Primary Notions that are intuitively understandable. Representations such as this are also possible, but unlikely, for complex subject matters. Thus, in general, we will find Definitions, Suppositions and Results, and only these Statements, in the set of Statements.

Let R_1 be an Axiomatically Structured Representation for the same subject matter for which R is also a Representation. Let R_1 have as components L_1, PS_1, S_1, PT_1, ST_1, F_1, $\vdash_1$ and $\models_1$.

Definition 2. The Representation R_1 is said to be a Syntactic Simplification of R just in case the following eight conditions hold.

(1) R and R_1 use the same Primitive Terms.

(2) R_1 uses fewer Defined Terms than R and the Defined Terms in R_1 also appear in R.

(3) R and R_1 use the same underlying language. L^1_1 differs from L^1 by eliminating from the lexicon of L^1 Secondary Terms in R but not in R_1 added to L when L^1 was formed. All special transformation rules for these terms added to the grammar for L are also dropped.

(4) There is a correspondence associating each Statement in PS with a Statement in PS_1.

(5) Every Statement in S, excluding Definitions of Secondary Terms not appearing in ST_1 but in ST, is associated with a unique Statement in S_1. This association takes on the values assigned by the correspondence from condition 4.

(6) The association in condition 5 associates the Definition of a Term in R with its Definition in R_1.

(7) The association in condition 5 associates Suppositions in R with Suppositions in R_1.

(8) Every Statement in PS is provably equivalent to its association in PS_1 arising from condition 4.

Thus, if R_1 is a Syntactic Simplification of R, then R_1 and R express the same facts about the subject matter, but R_1 does so more parsimoniously. Also everything expressible in R_1 (PS_1) is expressible in R (PS). On the other hand,some things expressible in R are not expressible in R_1, since ST_1 is a proper subset of ST.

A Syntactic Reduction of a representation, R, for a subject matter is a Syntactic Simplification, R_1, of R such that Statements provable in R correspond under condition (4) above to Statements provable in R_1. In other words, if R_1 is a Syntactic Reduction of R, then R_1 is a Syntactic Simplification of R in which derivational aspects of Statements concerning the subject matter represented by R carry over to R_1.

Definition 3. An Axiomatically Structured Representation of a subject matter is said to be Primitively Based if it employs no Secondary Terms (Defined Notions).

Thus, R is Primitively Based if $ST=\emptyset$, the empty set. It can be shown that any Axiomatically Structured Representation of a subject matter is Syntactically Reducible to a Primitively Based Axiomatically Structured Representation of the same subject matter. Thus, given a Representation of a subject matter, we can always find another Representation of the subject matter that expresses the same facts about the subject matter, and which does not use Secondary Terms or Defined Notions. This is a strong sense of the notion of the eliminability of Secondary Terms or Defined Notions.

Theorem 2. Every Axiomatically Structured Representation is Syntactically Reducible to a Primitively Based Axiomatically Structured Representation.

Without pursuing the details, we can formulate other interesting notions that serve to relate Representations. The Representation, R_1, is a *Translation* of R if $L_1 \neq L$, but PT and PT_1 have the same number of elements, as do ST and ST_1. The Representation R_1 will be a *Conceptual Simplification* of R, if the number of required Primary Terms is reduced in going from R to R_1. That is, R_1 uses fewer Primary or undefined Notions to Represent the same subject matter as R. A Conceptual Simplification is a *Conceptual Reduction*, if the Simplification retains the derivational strength of the original Representation.

This concludes our discussion of eliminability and non-creativity. We now take up the question of the structure exhibited by an Axiomatically Structured Representation of a subject matter. One of the motivations behind this inquiry is to establish that Axiomatically Structured Representations exhibit another form of internal consistency that might be called structural consistency as opposed to what might be called definitional consistency as discussed above. The idea behind structural consistency is that Dependency (to be defined) should not be cyclic. Statements should *not* Depend upon themselves either Syntactically or Deductively. The following developments are intended to make these notions more precise.

Dependency can be seen to be based on the notions of Syntactic and Deductive Dependence. A Statement is *Syntactically Dependent* on a Term (Primary or Secondary) if there is a sequence of "used in the formulation of" relationships leading from the Term to the Statement. For example, if Term t is used to formulate Statement s, i.e., Fts, then s is Syntatically Dependent on t. As another example, suppose Term t is used to formulate the Definition of another Term t_1, and assume that t_1 is used to formulate s, i.e., Ftt^*_1 and Ft_1s, then s, is Syntactically Dependent on t. In the most general case we have the following situation.

Term t is used to formulate t_1*

Term t_1 is used to formulate t_2*

Term t_2 is used to formulate t_3*

etc.

Term t_n is used to formulate s.

Whence, by this chain of "used to formulate" relationships, s is Syntactically Dependent on t.

For formal reasons we will define Syntactic Dependency in such a way that t* is not Syntactically Dependent on t. On the surface this approach may seen strange but the following argument is intended to suggest that this approach is acceptable. Our interest in Syntactic Dependency is motivated in part from a desire to be able to stipulate the order in which subject matter related Statements should be introduced to the learner. We expect all Secondary Terms used to formulate a Statement to be Defined before the Statement is presented to the learner. Now to discover that a Secondary Term is used to formulate its own Definition provides us with no information relative to sequencing. We cannot Define a Secondary Term before we use it in its own Definition. The Definition itself is intended to be the first occurrence of the Term. Thus, anticipating that sequencing is to be based, in part, on Syntactic Dependency; and recognizing that the Definition of a Term is intended to be the Term's first use in the sequence of Statements; leads us to exclude Ftt* as a case of Syntactic Dependency.

Definition 4. A Statement, s, is Syntactically Dependent on t, either a Primitive or Secondary Term, if and only if

(1) s is not the Definition of t: $s \neq t^*$, and

(2) there is a sequence, possibly empty, of Secondary Terms $t_1, \ldots, t_n$ such that

$$F_{tt_1*} \mathbin{\&} F_{t_1t_2*} \mathbin{\&} \ldots \mathbin{\&} F_{t_{n-1}t_n*} \mathbin{\&} F_{t_n s}$$

holds.

Definition 4 applies to any Primitive Term, any Secondary Term and any Statement. Thus, if t is a Primitive Term and s is any Statement such that Fts holds, then s is Syntactically Dependent on t. Also, if t is any Secondary Term and s is any Statement such that $s \neq t^*$ and Fts holds, then s is Syntactically Dependent on t. Definition 4 implies that t* is not Syntactically Dependent on t for t in ST.

Theorem 3. The Definition of a Secondary Term is not Syntactically Dependent on the Term Defined in the Definition.

We expect the "used in the formulation of" relationship to be non-cyclic. That is, it would be incorrect to use one Defined Notion to define a second; the second being used to define a third; the third a fourth; and so on to an n^{th} Notion used to define the first. We eliminate definitional circularity such as this with an assumption.

Assumption 11. The "used in the formulation of" relationship is non-cyclic in that there are no sequences of the following form where $n \geq 1$, $t, t_1, \ldots,$ tn are all members of ST.

$$F_{tt_1^*} \mathbin{\&} F_{t_1t_2^*} \mathbin{\&} \ldots \mathbin{\&} F_{t_{n-1}t_n^*} \mathbin{\&} F_{t_n t^*}$$

Notice in Assumption 11 that we do not need to consider the possibility of t in PT since such a Term does not have a Definition.

Theorem 4. Syntactic Dependency exhibits a form of transitivity. For all t in ST, for all t_1 in PT∪ST and for all s in S; if s is Syntactically Dependent on t and if t^* is Syntactically Dependent on t_1, then s is Syntactically Dependent on t_1.

To establish Theorem 4 assume that s is Syntactically Dependent on t and t^* is Syntactically Dependent on t_1. This means we have both

$$Ftx_1^* \ \& \ \ldots \ \& \ Fx_n^*s$$

and

$$Ft_1y_1^* \ \& \ \ldots \ \& \ Fy_mt^*.$$

We obviously have the required sequence from t_1 to s rendering s Syntactically Dependent on t_1. Assumption 11 assures us that Theorem 4 does not lead to a contradiction of Theorem 3.

We expect Syntactic Dependency to be non-cyclical in that we do not want an element to Syntactically Depend on itself either directly or through a chain of Syntactic Dependency relationships. Cyclic Syntactic Dependency would be a chain of the following form

t_1^* is Syntactically Dependent on t

t_2^* is Syntactically Dependent on t_1

.
.
.

t_n^* is Syntactically Dependent on t_{n-1}

t^* is Syntactically Dependent on tn

where t is in ST and $n \geq 1$. We do not need to consider the possibility of $t \epsilon PT$ since such a t does not have a Definition. Nor do we need to consider the case where t^* is a Statement other than a Definition: Statements cannot be used to formulate other Statements. Also, we need only consider the case where $n \geq 1$ since t^* is not Syntactically Dependent on t by Theorem 3. Sequences of the above form are eliminated by Theorems 3 and 4 in conjunction.

Theorem 5. Syntactic Dependency is non-cyclic.

Our next definition introduces the notion of Deductive Dependency which in based on the relationship "$\vdash$". Suppose we have two Statements s, s_1 and a subset, A, of S such that

$$\begin{array}{ll} & s_1 \epsilon A \\ \text{and} & A \vdash s \end{array}$$

both hold. In such a case it would be natural to say that s is Deductively Dependent on s_1. The more general case arises when we have the following conditions.

$$\begin{array}{ll} s_1 \epsilon A_1 & A_1 \vdash s_2 \\ s_2 \epsilon A_2 & A_2 \vdash s_3 \\ \vdots & \\ s_{n-1} \epsilon A_{n-1} & A_{n-1} \vdash s_n \\ s_n \epsilon A_n & A_n \vdash s \end{array}$$

Again, we would assert in this case that s is Deductively Dependent on s_1 since we have a sequence of "is proved from" relationships leading from s_1 to s.

Definition 5. A Statement s is Deductively Dependent on a Statement s_1 if and only if there is a sequence (possibly empty) of Statements $y_1, \ldots, y_{n-1}$ and a sequence of sets $A_1, \ldots, A_n$ ($n \geq 1$) such that the following relationships hold:

$$A_1 \vdash y_1,\ A_2 \vdash y_2,\ \ldots,\ A_{n-1} \vdash y_{n-1},\ A_n \vdash s$$

where s_1 is in A_1 and y_i is in A_{i+1} for $i=1, \ldots, n-1$.

We expect Deductive Dependency to be non-cyclic. That is, it would be incorrect to have one Statement included in a set used to prove a second Statement; the second in a set used to prove a third; and so on up to an n^{th} Statement in a set used to prove the first. This classifical form of circular reasoning must be avoided. In the simplest case we want to eliminate the following possibility.

$s \varepsilon A_1$ $\qquad A_1 \vdash s_1$

$s_1 \varepsilon A_2$ $\qquad A_2 \vdash s$

In this case we have both

s_1 is Deductively Dependent on s

and s is Deductively Dependent on s_1.

Together these statements indicate a symmetry for Deductive Dependency. Since symmetry of this type is to be avoided, we rule it out with our last assumption.

Assumption 12. Deductive Dependency is an asymmetric relationship.

Paranthetically, assumption 12 is the last assumption promised earlier. With this assumption we now have introduced all necessary assumptions to formally define the notions introduced above such as an Axiomatically Structured Representation. These notions and results based on them were introduced informally. The informal treatment is sufficient for our purposes. Whence, we continue our discussion of structure.

From assumption 12 we can prove that Deductive Dependency is irreflexive: Theorem 6 below. That is, s is not Deductively Dependent on itself. Deductive Dependency is also transitive as stated also in Theorem 6. This fact follows directly from the definition of Deductive Dependency.

Theorem 6. Deductive Dependency is an irreflexive, transitive relationship.

We can now show that Deductive Dependency is non-cyclic. A cyclic chain of Deductive Dependencies would result in the following conditions.

s_1 is Deductively Dependent on s.

s_2 is Deductively Dependent on s_1.

.
.
.

s_n is Deductively Dependent on s_{n-1}.

s is Deductively Dependent on s_n.

From Theorem 6 (transitivity) these conditions imply that s is Deductively Dependent on s, contradicting the fact that Deductive Dependency is irreflexive (Theorem 6). Thus, we have the following theorem.

Theorem 7. Deductive Dependency is non-cyclic.

The notions of Syntactic and Deductive Dependency can be combined to form the concept of Dependency. If s is Syntactically Dependent on x or if

47

s is Deductively Dependent on x, then s is Dependent on x. In the first of these cases, s is a Statement and x is a Primary or Secondary Term. In the second case, s is a Statement and x is a Statement. Notice also that it may be the case that s is Deductively Dependent on y and y is Syntactically Dependent on x. In this eventuality we would assert also that s is Dependent on x, where s and y are Statements and x is a Primary or Secondary Term. The case where s is Syntactically Dependent on y and y is Deductively Dependent on x cannot arise since y cannot be both a Primary or Secondary Term, and a Statement--Axiom 1, Part (4). The formal definition of Dependency is as follows.

<ins>Definition 6.</ins> A Statement s is Dependent on x if and only if one of the following conditions hold.

(1) s is Syntactically Dependent on x, and x is a Primary or Secondary Term.

(2) s is Deductively Dependent on x, and x is a Statement.

(3) There is a Statement s_1 such that s is Deductively Dependent on s_1 and s_1 is Syntactically Dependent on x where x is a Primary or Secondary Term.

Dependency will be asymmetric, transitive and non-cyclic as stated in the next three theorems.

<ins>Theorem 8.</ins> Dependency is an asymmetric relationship.

<ins>Theorem 9.</ins> Dependency is a transitive relationship.

<ins>Theorem 10.</ins> Dependency is a non-cyclic relationship.

To prove Theorem 8, let s be Dependent on x, which means s is in S and either x is in S or x is in PT∪ST. If x is in S we know that s must be Deductively Dependent on x. By Axiom 12 x is not Deductively Dependent on s. Also no element (including x) can be Syntactically Dependent on s, so x cannot be Dependent on s. Finally, if x is in PT∪ST then x cannot be Dependent on any elements, including s. Therefore, if s is Dependent on x, then x is not Dependent on s, establishing Theorem 8.

For Theorem 9, suppose s_2 is Dependent on s_1 and s_1 is Dependent on s. Thus, s_2 is in S and s_1 is in S which implies s_2 is Deductively Dependent on s_1. If s_1 is Deductively Dependent on s, then s_2 is Deductively Dependent on s by virtue of Theorem 6. Whence, s_2 is Dependent on s. If s_1 is Syntactically Dependent on s, then s_2 is Dependent on s by condition (3) of the definition of Dependency. The last possibility is that s_1 is Deductively Dependent on some element s_3 and s_3 is Syntactically Dependent on s. In this case s_2 is Deductively Dependent on s_3, and applying condition (3) of the definition of Dependency, yields s_2 is Dependent on s.

To prove Theorem 10, assume the following conditions hold.

s is Dependent on s_1
s_1 is Dependent on s_2
.
.
.
.
s_{n-1} is Dependent on s_n
s_n is Dependent on s.

With these conditions Theorem 9 implies that s is Dependent on s. But Theorem 8 entails the irreflexitivity of Dependency, eliminating this possibility.

The notion of Dependency is precisely the notion required for constraining the order of instructional transmission. Suppose a Statement s is Dependent on x where s is either a Term (Primary or Secondary) or a Statement. There are three possible ways in which this Dependency may arise. The first case is for s to be Syntactically Dependent on x where x is a Term. In this case, the definition of x,x* should precede s. Also, since s uses x in its formulation, a learner must be familiar with the Notion represented by x before the learner can understand s. Thus, instruction on x, or on the Notion x represents, should be scheduled before instruction on s.

In the second case s is Deductively Dependent on another Statement x making s Dependent on x. In this eventuality we desire to have x precede s, instructionally. That is, a learner about to receive instruction concerning s should have prior instruction concerning x. Otherwise it is not clear how the learner will come to comprehend s, since it has been determined that s Depends on x.

Finally, s may be Deductively Dependent on Statement s_1 and s_1 may be Syntactically Dependent on Term x. Again, we want x*, the Definition of x, to precede s_1 and s_1 to precede s. Thus, x* will be expected to precede s. A learner in this case requires instruction concerning both x and s_1 before receiving instruction concerning s.

Therefore, it seems that the concept of Dependency will serve the purpose of constraining the sequencing of instructional material during instruction. The details of this process are beyond the scope of this Chapter. Briefly, it does seem that the Instructional Process (Figure 1) can be viewed as a dialogue between the instructional agent and learner. The Dependency relationship constrains the order in which Statements may be introduced in the dialogue. In fact, given these restraints, it will follow that the dialogue itself exhibits the properties of an Axiomatically Structured Theory! The development of the details concerning these notions, for the present, must be postponed.

While Definition 6 provides us with the appropriate characterization of Dependency, Definition 6 does not provide much help in identifying all Dependency relationships within an Axiomatically Structured Representation. A matrix method based on graph theoretical principles has been developed by Harary et al (1965) that can be adopted to provide a mechanical procedure for identifying all Dependency relationships. Briefly, this method assumes the existence of n elements that are interrelated by 1 or more non-cyclic relationships. An adjacency matrix is developed for each relationship indicating by a row and column entry that the row element is related directly to the column element. A matrix entry of 1 serves to indicate the existence of this direct (adjacent) relationship. The matrices are then added element by element. The resultant matrix is then reduced to 0's and 1's by setting an entry to 1 just in case the sum for that row and column is greater than or equal to 1. Thus, the resultant matrix has a row, column entry equal to 1 just in case at least one of the original relationships relates the row element to the column element. Notice that from the reduced resultant matrix we can neither tell what relationship relates the elements, nor can we tell how many relationships relate the elements.

When the matrices are based on the Syntactic and Deductive Dependency relationships, the reduced resultant matrix shows all adjacent Dependency relationships that exist between the n elements. This matrix is an $n \times n$ matrix and can be squared. The square of the matrix indicates for any two elements the number of 2-step, or 2-link, Dependency chains that connect the elements. This matrix can be reduced to 0's and 1's by the procedure used before. In general, the m^{th} power of the original reduced matrix shows for each two elements the number of m-step, or m-link, Dependency chains that connect the elements.

For some m it can be shown that the $m+1^{st}$ power of the original reduced matrix must be the zero matrix. If we add all the powers of the original reduced Dependency matrix up to and including m, the resulting matrix indicates all Dependency relationships. Suppose we reduce this final matrix to 0's and 1's as before. We can show that in this matrix a row,column entry is 1 just in case the row element is related to the column element by a sequence of the original relationships. The matrix does not tell us what relationships make up this sequence; it does not tell us how many relationships are in the chain; and it does not tell us the order in which the relationships appear in the chain. All the matrix tells us is that there is some chain of relationships from the row element to the column element, i.e. the column element is Dependent on the row element.

We now formalize the procedure outlined above. We begin with the formal definition of an adjacency matrix.

Definition 7. The adjacency matrix for an arbitrary relationship, r, defined between n elements, is the n×n square matrix $A(r)=[a_{ij}]$ where $a_{ij}=1$ if element i is related to element j by r, and $a_{ij}=0$, otherwise.

We have two relationships on which Dependency is based: "used to formulate" and "is proved from". We will use these relationships to develop two n×n adjacency matrices to be used in the procedure sketched above. First consider the "used to formulate" relationship. Consider the set made up of elements from PT and from S (i.e. $PT \cup S$); and let n be the number of elements in this set. Order these n elements in some way. As to be discussed below we do not distinguish Secondary Terms from the Definitions in this list.

Definition 8. The Adjacency Matrix for Syntactic Dependency is defined as the n×n matrix $A(SD)=[a_{ij}]$ which is constructed by the following three rules.

(1) If element i is in PT, and element j is in S, then $a_{ij}=1$ if element i is used to formulate element j.

(2) If element i is a Definition and element j is in S, and if $i \neq j$, then $a_{ij}=1$ if the Secondary Term defined in element i is used to formulate element j.

(3) All other entries are zero.

Of the three rules in Definition 8 only rule (2) requires some explanation. As mentioned above we do not distinguish Secondary Terms and their Definitions in the Adjacency Matrix for Syntactic Dependency. That is, we have simply listed elements from PT and S, but we have not included elements from ST since the Definitions are already listed. Thus, when we consider a row element that is a Definition we do not ask if that element is used to formulate a member of S. Definitions are Statements and, as such, are not used to formulate other Statements. What we must ask is whether or not the Secondary Term being Defined is used to formulate other Statements, as we have done in rule (2).

A simple example will show that this procedure is the correct procedure. Consider three Definitions of three Secondary Terms where the first is used to formulate the Definition of the second and the second is used to formulate the Definition of the third. Definition 8 leads in this case to the following matrix (ignoring Primitive Terms and other possible Statements).

$$\begin{array}{c|ccc} & D1 & D2 & D3 \\ \hline D1 & 0 & 1 & 0 \\ D2 & 0 & 0 & 1 \\ D3 & 0 & 0 & 0 \end{array}$$

Squaring this matrix yields

$$\begin{array}{c|ccc|} & D1 & D2 & D3 \\ D1 & 0 & 0 & 1 \\ D2 & 0 & 0 & 0 \\ D3 & 0 & 0 & 0 \end{array}$$

which shows (by the above outlined procedure) that the third Definition is Syntactically Dependent on the first Secondary Term as expected.

Suppose contrary to Definition 8 we distinguish between Secondary Terms and Definitions. This would lead to the following adjacency matrix showing all instances of the "used in the formulation of" relationships.

$$\begin{array}{c|cccccc|} & ST1 & ST2 & ST3 & D1 & D2 & D3 \\ ST1 & 0 & 0 & 0 & 1 & 1 & 0 \\ ST2 & 0 & 0 & 0 & 0 & 1 & 1 \\ ST3 & 0 & 0 & 0 & 0 & 0 & 1 \\ D1 & 0 & 0 & 0 & 0 & 0 & 0 \\ D2 & 0 & 0 & 0 & 0 & 0 & 0 \\ D3 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

The square of this matrix is the zero matrix or the matrix containing only zeroes and does not show that Definition 3 is Syntactically Dependent on the first Secondary Term. Whence, this approach is in error and, appropriately, Definition 8 avoids this pitfall.

By rule (2), and in the above example, we do not indicate in the Syntactic Dependency Adjacency Matrix that a Secondary Term is used to formulate its own Definition. By definition we know that a Definition is not Syntactically Dependent on the Secondary Term it defines. Thus, there are no ones along the diagonal of the matrix A(SD). In any event, if our interest in Dependency is motivated by a desire, for example, to insure that all elements an element depends upon are explained first, then to discover that a Definition depends upon the Secondary Term it defines is of little value. The Definition is intended, after all, to introduce the Secondary Term into the theory. Thus, knowing that this particular Dependency exists will not result in the prior introduction of other Statements. This is in contrast to a case in which a Definition of a Secondary Term Depends on other Secondary Terms. In this case we would make sure the other Terms are Defined first before proceeding to the new Definition.

From a procedural point of view we must not indicate that a Secondary Term is used to formulate its own Definition. To do so means that a 1 will be entered along the diagonal of the adjacency matrix. This means that the relationship being represented is cyclic. In other words raising the matrix to successive powers never results in the zero matrix and, thus, the procedure to be used will not be applicable to the problem at hand. Whence, Definition 8 results in an adjacency matrix for a non-cyclic relationship (Theorem 5)--as required.

The adjacency matrix for Deductive Dependency is defined in terms of the same n elements, arranged in the same order, as used to define the Adjacency Matrix for Syntactic Dependency.

Definition 9. The Adjacency Matrix for Deductive Dependency is defined as the $n \times n$ matrix $A(DD)=[b_{ij}]$ which is constructed by the following rules.

(1) If element j in $PT \cup S$ is derived from a set of Statements that includes element i in $PT \cup S$ then $b_{ij}=1$.

(2) All other entries are zero.

The meaning of Definition 9 is self evident. Notice, however, that the rows and columns in A(DD) correspond to the rows and columns in A(SD). Also notice that there are no diagonal entries in A(SD) since Deductive Dependency is irreflexive.

For convenience we introduce the notion of Boolean Arithmetic. The only difference between Boolean Arithmetic and non-Boolean Arithmetic is that in Boolean Arithmetic $1+1=1$. Following Harary, et al (1965) we denote Boolean Arithmetic by writing $(n+m)\#$. Thus, $(1+1)\#=2\#=1$ and, in general, it follows that if $n\geq 1$, then $n\#=1$.

Definition 10. For all positive integers n,m $(n+m)\#$ is the Boolean Sum of n and m where $(n+m)\#=0$ if $n+m=0$ and $(n+m)\#=1$ if $n+m\geq 1$.

Definition 11. The Adjacency Matrix for the Dependency relationship, A(D), is defined as

$$A(D)=[A(SD)+A(DD)]\#.$$

To form the Adjacency Matrix for the Dependency relationship the Adjacency Matrices for Syntactic and Deductive Dependency are constructed first. These Adjacency Matrices are then Boolean added element by element. This is normal matrix addition except that the individual sums are Boolean sums so that each entry in A(D) is either a 1 or 0. It follows that each element, c_{ij}, of A(D) is such that

$c_{ij} = 1$ if the j^{th} element in $PT \cup S$ is either directly Syntactically Dependent or directly Deductively Dependent on the i^{th} element in $PT \cup S$.

and

$c_{ij} = 0$ if the j^{th} element in $PT \cup S$ is neither directly Syntactically Dependent nor directly Deductively Dependent on the i^{th} element in $PT \cup S$.

Whence, A(D) depicts the Dependency relationship to the extent of showing all dependency chains of length 1. The matrix A(D) does not necessarily depict all the Dependency relationships since Dependency may be established through relational chains of length greater than 1. The next theorem provides the rationale needed to generate the complete dependency matrix.

Theorem 11. (Harary, et al, 1965). In $A(D)^{\ell}$ ($A(D)$ raised to the ℓ^{th} power) the i,j^{th} entry is the number of chains of Dependency relationships of length ℓ leading from the i^{th} element in $PT \cup S$ to the j^{th} element in $PT \cup S$.

It can be shown that for some $m>1$, $A(D)^{m+1}=[0]$, the matrix containing only zeros. This follows since there are no loops in the Dependency relation. Thus, since the number of elements in $PT \cup S$ is finite, there is an upper limit on the length of possible Dependency relationship sequences leading from one element of $PT \cup S$ to another. We call this maximum m the Maximal Length Dependency Chain.

Definition 12. The Maximal Length Dependency Chain for an Axiomatically Structured Representation is defined as the minimal m such that $A(D)^{m+1}=[0]$.

Definition 13. The Dependency Matrix for an Axiomatically Structured Representation, R, is defined as

$$D_R = [A(D)^1 + A(D)^2 + \dots + A(D)^m]\#$$

where m is the Maximal Length Dependency Chain for R.

When there is no confusion about which Axiomatically Structured Representation, R, is being discussed we will write D in lieu of D_R.

Suppose D is the Dependency Matrix for an Axiomatically Structured Representation and let x,y be elements in $PT \cup S$. Within D, x will correspond to some row and y will correspond to some column. We will write a_{xy} to indicate the entry in D consisting of the row and column corresponding to x and y, respectively. If x is represented by the i^{th} row and j is represented by the j^{th} column in D, then $a_{xy}=a_{ij}$.

Definition 14. a_{xy} denotes the entry in D consisting of the row and column corresponding to x and y respectively for x,y in $PT \cup S$.

Theorem 12. Let x,y be elements in $PT \cup S$. y is Dependent on x if and only if $a_{xy}=1$.

If y is Dependent on x then there will be a sequence of Dependency relationships leading from x to y. Such a sequence will be composed of say ℓ steps or links where $\ell \leq m$. Thus, the entry representing the x to y Dependency relationship in the ℓ^{th} power of A(D) will be 1. When the Dependency Matrix is formed by Boolean adding the powers of A(D) up to $m \geq \ell$ this entry will carry over making $a_{xy}=1$. Conversely, if $a_{xy}=1$, then there will be some power of A(D), say ℓ, where $\ell \leq m$ and the row, column entry corresponding to x and y will be non-zero. Thus there must be at least one Dependency sequence of length ℓ from x to y. That is, y is Dependent on x.

Theorem 13. Let t be in ST and let s be in S. s is Dependent on t if and only if $a_{t*s}=1$.

To prove Theorem 13 let x be in ST and s be in S. Suppose first that s is Dependent on t. Rule (2) in Definition 8 assures us that the row entry in the Syntactic Dependency Adjacency Matrix corresponding to t* will contain a 1 in some column that subsequently links t* to s via a sequence of Dependency links. This sequence will eventually result in a 1 being entered in the row,column corresponding to t*,s for some power of the Adjacency Matrix for Dependency. When A(D) is formed this entry will carry over causing a_{t*s} to be set to 1.

Conversely suppose $a_{t*s}=1$. This entry is caused by a 1 entered in some power of the Adjacency Matrix for Dependency. Whence, there is an ℓ step chain of Dependencies from t* to s where $\ell \leq m$. In other words, s is Dependent on t. Rule (2) in Definition 8 again is used to ensure us that t* is linked to some element in the Adjacency Matrix for Syntactic Dependency; and this other element, if not s itself, provides the Dependency links to s.

Theorems 12 and 13, and the concepts on which they are based, provide a direct method for developing the complete Dependency relationship for an Axiomatically Structured Theory.

We now turn to the development of an example of these notions. As an example we choose elementary photography. We do not intend to develop a complete course in photography but simply to indicate how to apply the foregoing theoretical developments. Our example will be a reformulation of the photography example presented by Stelzer and Kingsley (1974b). Stelzer and Kingsley's analysis of the photography course is intended to show that an actual course can be recast in terms of the theoretical axiomatic paradigm they develop. Our emphasis is slightly different in that we assume an author is confronted with the original problem of developing such a course. Thus, Stelzer and Kingsley start from a given course for a subject matter, but we start from the subject matter and have as a goal a course.

An author may begin to summarize informally the photography subject matter in the following manner. Photography is a three-step process: Exposure, Processing, and Printing. Exposure consists of light rays reflected from a subject passing through the lens, diaphragm and shutter to form a latent image in the film. The film is an emulsion consisting of silver halide grains in a gelatin. The camera is the device containing the lens, diaphragm, shutter and film. Silver halide grains in the film have varying sensitivity to light. Emulsion speed is a measure of how fast the emulsion material reacts to light. Emulsion film speeds are measured numerically so that higher numbered emulsions are more sensitive. Silver halide grains in the film also tend to clump together. This clumping determines the size of the grain in the film. Higher speed films tend to have larger grains. Inherent grain size is the tendency of a particular film to produce a grain of a certain size.

Granular image structure or grain size also affects the sharpness of the image. Small regular emulsion grains yield sharper images. Resolving power is the inherent property of an emulsion to reproduce fine detail. Resolving power is numerically measured by expressing the number of lines per millimeter that can be distinguished individually in the photographic image. The smaller the distance between individual lines, the greater the number of lines per millimeter, the greater the resolving power, and the sharper the image detail.

Darkness is formed in the film emulsion as a result of exposure. The darkness, or light-stopping ability, is referred to as density. Contrast is the difference between high- and low-density areas. Bright subject areas cause heavy density, while dark subject areas cause light density.

Processing involves the use of developer to create visible black metallic silver from the exposed emulsion. The film goes through a rinse bath, a hypo solution or fixer, and is washed. As a result, the latent image becomes a visible negative image. Drying is the last processing step.

Printing involves passing transmitted light through the negative onto a paper emulsion. Print processing involves development, fixing, washing, and drying.

The Processing and Printing notions can be traced down further, but we will not do so herein.

Given a subject-matter summary such as this, the author is in a position to develop an Axiomatically Structured Representation for the subject matter. The first step is to select the language L. The above summary being in English leads naturally to the selection of English for L. But L must clearly be augmented with some basic notions from chemistry and physics. This is an important point and must not be overlooked. A natural language such as English is often, and for complex subject matters even usually, not sufficient for L. Other specialized languages for other subject matters must be subsumed as part of the available language framework within which instruction is to take place. In our example, if some notions concerning basic physics and chemistry are not assumed to be available, i.e., within L, then instruction concerning these notions must be offered rendering them part of the instructional subject matter.

Analysis of the above photography summary leads to the identification of ten Primary Notions represented by the following listed ten Primary Terms. We have labeled these Terms for later reference.

PT1: light rays

PT2: reflection

PT3: subject

PT4: silver halide grains

PT5: gelatin

PT6: sensitivity to light

PT7: inherent grain size

PT8: granular image structure

PT9: visible black metallic silver

PT10: transmitted light

To reiterate our point concerning the identification of the underlying language L, consider PT1 which represents the Primary Notion of light rays. Light rays are a concept from elementary physics. Since this Notion is assumed to be primary and, whence, intuitively understandable, it must be that the term appears in the lexicon of L. That is, the instructor is willing to assume the learner is familiar with the concept and familiar with the syntax of the term. Thus, L is seen to include elementary physical and chemical concepts.

On the basis of these 10 Primary Notions (and others not given in the above summary) we can define 30 Secondary Notions. These Notions are as follows where we have labeled the Terms, and provided the intended Definitions of the Terms. In each case the Defined Term is underlined. The Statements defining the Terms are labeled at the end of the Sentences as D1-D30. That is, for example, ST1 labels a Secondary Term, Lens, while D1 labels the Term's Definition--a Statement.

ST1: A lens is a device for collecting and focusing light rays reflected from the subject: D1.

ST2: A diaphragm is a device controlling the amount of light that passes through the lens: D2.

ST3: Emulsion is the light-sensitive portion of photographic material: D3.

ST4: Emulsion grain is the microscopic light-sensitive particles in an emulsion: D4.

ST5: Film is an emulsion consisting of silver halide grains in a gelatin: D5.

ST6: A shutter is a device that controls the length of time light from the lens is allowed to strike the film: D6.

ST7: The latent image is the image of the subject formed in an emulsion: D7.

ST8: A camera is a device consisting of lens, diaphragm, shutter, and film holder: D8.

ST9. The emulsion speed is a measure of how fast the emulsion reacts to light: D9.

ST10: The resolving power is the inherent property of the emulsion to reproduce fine detail; measured numerically; depends on inherent grain size and granular image structure: D10.

ST11: *Film exposure* is the process of forming the latent image in the film from light rays reflected from the subject into the camera: D11.

ST12: *Density* is darkness in the film emulsion resulting from exposure: D12.

ST13: *High density* means dark areas: D13.

ST14: *Low density* means lighter areas: D14.

ST15: *Contrast* is the difference between high- and low-density areas. D15.

ST16: *Film development* is a chemical process used to create a visible blank metallic silver image from the exposed film emulsion: D16.

ST17: A *film rinse bath* is used to wash developer from film after film development: D17.

ST18: *Film hypo solution* or *fixer* is a chemical used to dissolve remaining film emulsion after development: D18.

ST19: *Film wash* consists of washing chemicals from film after fixing: D19.

ST20: A *visible negative image* is a film image that results from development, rinsing, fixing, and washing exposed film: D20.

ST21: *Film drying* is the last step in developing film: D21.

ST22: *Processing* is the process of forming a visible negative image from the exposed film by developing, rinsing, fixing, washing, and drying: D22.

ST23: *Paper emulsion* is an emulsion contained in paper: D23.

ST24: Print exposure is transmitting light through the visible negative image in order to create a latent image in the paper emulsion: D24.

ST25: Print development is a chemical process used to create a visible image in the exposed print: D25.

ST26: Print fixing means dissolving unused emulsion in the paper: D26

ST27: Print washing is washing the visible image print after fixing: D27.

ST28: Print drying is drying the print after washing: D28.

ST29: Printing is the process consisting of print exposure, print development, print fixing, print washing, and print drying: D29.

ST30: The print is the final product that results from exposure, processing, and printing: D30.

The set of Statements, S, will consist of these 30 Definitions, and the following 7 Suppositions and 12 Results. We have selected these principles merely for illustrative purposes from among the many that we might have selected. We hasten to add that there may be slight gaps in the Definitions listed above in that eliminability, Axiom 9, may not be satisfied completely. This is the case simply because we are not attempting to develop the complete Axiomatic Representation for photography herein for obvious reasons.

The 7 Suppositions are labeled S1-S7 and the 12 Results are labeled R1-R12.

S1: Silver halide grains in film emulsion have varying sensitivity to light that can be numerically measured.

S2: Silver halide grains in film tend to clump together determining grain size in the film.

S3: Small regular emulsion grains yield sharper images.

S4: Darkness is formed in the film emulsion as a result of exposure.

S5: Bright subject areas cause heavy density, while light subject areas cause low density.

S6: Darkness is formed in the paper emulsion as a result of print exposure.

S7: High-density film areas result in less exposed paper emulsion, while lower-density film areas cause more exposure in the paper emulsion.

R1: Higher numbered film emulsions have greater light sensitivity.

R2: Higher speed film emulsion tends to have larger grains.

R3: The greater the resolving power the sharper the image detail.

R4: Light subject areas are dark in the film, but dark subject areas are light in the film.

R5: Since printing is on white paper, dark (high density) film areas remain white, but light (low density film areas) become dark.

R6: Light subject areas are light in the print and dark subject areas are dark in the print.

R7: Thus, the print is an image of the subject.

R8: Light that gives brilliant subject highlights and dark black subject shadows produces higher-contrast visible negative images.

R9: High-contrast subjects produce higher-contrast negative images, regardless of the type of film being used.

R10: Negative-image contrast varies directly with development time.

R11: Since film emulsion is light-sensitive before processing, laboratory light must be controlled during processing.

R12: Since paper emulsion is light-sensitive before print fixing, laboratory light must be controlled during printing.

Not every Notion used in the Secondary Notions, Suppositions, and Results has been sorted out and included in the list of Primary and Secondary Notions. For example, R11 and R12 make use of the concept of controlling light, which has not been introduced or explained. This could be done, but for our purposes this effort does not seem justified. The listing of the 49 Statements for the photo course should be inclusive enough to demonstrate how to begin to apply the theoretical model described earlier.

The above Statements are taken almost verbatim from Stelzer and Kingsley's example. In effect, all that has been added herein is that every Secondary Term and its Definitional Statement has been distinguished clearly and labeled accordingly. Thus, we have in the present example 40 Terms: Ten Primary and thirty Secondary. We have also 49 Statements: Thirty Definitions, seven Suppositions and twelve Results. Thus, we have a total of 89 elements in the photography Representation. The figure 89 results because we distinguish Terms and their Definitions. Stelzer and Kingsley do not distinguish Terms from their Definitions; thus they have only 59 constituents.

The next task is to describe the F and $\vdash$ relations used to interrelate these 89 elements. First, consider F, the "used in the formulation of" relation. A tabular summary of the F relationship for every Sentence is given in Table 1. The list of elements in the row for any of these entries is a list of all Terms that are used in the formulation of that element. For example, consider D30, the Definition of the Notion of a print. From Table 1 we see that:

(1) ST11 is used in the formulation of D30.

(2) ST22 is used in the formulation of D30.

(3) ST29 is used in the formulation of D30.

(4) ST30 is used in the formulation of D30.

As another example notice that Result 12 is formulated in terms of PT1, PT6, ST23, ST26 and ST29. Now strictly speaking 'light rays', i.e., the Term "light rays", does not appear in R12. But is is obvious that small incongruencies such as these can be expected in a partial example such as we are developing. These incongruencies would not appear in a more systematic and complete development. In this case, we might introduce the term "laboratory light", defined as a collection of light rays in the laboratory. In any event R12 surely is formulated in terms of the notion of light rays, PT1, as indicated in Table 1. Notice also that every Secondary Term is used to formulate its own Definition as indicated in Table 1. That is, for example, ST30 is used to formulate D30.

Table 2 presents a tabular summary of the "used to prove" relationship, $\vdash$. The interpretation of Table 2 is entirely similar to the interpretation of Table 1. For example, the Definition of Secondary Term ST10, D10, and Supposition 3, S3, are used to prove Result 3, R3.

Because a complete analysis of the photo course has not been provided, Table 2 is somewhat incomplete. That is, it may not be possible to prove completely a given Result on the basis of the elements listed. This follows, since the argument given to prove a Result may depend upon some elements

Table 1: TABULAR REPRESENTATION OF THE "USED IN THE FORMULATION OF" RELATION FOR THE PHOTOGRAPHY REPRESENTATION

D1	PT1, PT2, PT3, ST1
D2	PT1, ST1, ST2
D3	PT6, ST3
D4	PT6, ST3, ST4
D5	PT4, PT5, ST3, ST5
D6	PT1, ST1, ST5, ST6
D7	PT3, ST3, ST7
D8	ST1, ST2, ST5, ST6, ST8
D9	PT1, ST3, ST9
D10	PT7, PT8, ST3, ST10
D11	PT1, PT2, PT3, ST5, ST7, ST8, ST11
D12	ST3, ST5, ST11, ST12
D13	ST12, ST13
D14	ST12, ST14
D15	ST13, ST14, ST15
D16	PT9, ST3, ST5, ST11, ST16
D17	ST5, ST16, ST17
D18	ST3, ST16, ST18
D19	ST5, ST18, ST19
D20	ST5, ST16, ST17, ST18, ST19, ST20
D21	ST5, ST21
D22	ST5, ST11, ST16, ST17, ST18, ST19, ST20, ST21, ST22
D23	ST3, ST23
D24	PT10, ST7, ST20, ST23, ST24
D25	ST24, ST25
D26	ST3, ST23, ST26
D27	ST26, ST27
D28	ST27, ST28
D29	ST24, ST25, ST26, ST27, ST28, ST29
D30	ST11, ST,22, ST29, ST30
S1	PT4, PT6, ST3, ST5
S2	PT4, ST4, ST5
S3	PT7, PT8, ST4
S4	ST3, ST5, ST11
S5	PT3, ST13, ST14
S6	ST23, ST24
S7	ST5, ST13, ST14, ST23, ST24
R1	PT6, ST3, ST5
R2	ST3, ST4, ST5, ST9
R3	ST10
R4	PT3, ST5
R5	ST5, ST13, ST14, ST29
R6	PT3, ST30
R7	PT3, ST30
R8	PT1, PT3, ST15, ST20
R9	ST5, ST15, ST20
R10	ST15, ST16, ST20
R11	PT1, PT6, ST3, ST5, ST22
R12	PT1, PT6, ST23, ST26, ST29

Table 2: TABULAR REPRESENTATION OF THE "USED TO PROVE" RELATIONSHIP FOR THE PHOTOGRAPHY REPRESENTATION

R1	D3, D5, S1
R2	D3, D4, D5, D9, S1, S2, R1
R3	D10, S3
R4	D5, S4, S5
R5	D5, D13, D14, D29, S6, S7
R6	D30, R4, R5
R7	D30, R6
R8	D15, D20, R4
R9	D5, D15, D20, R8
R10	D15, D16, D20
R11	D3, D5, D22, R4
R12	D23, D26, D29, R5

that have not been singled out for attention herein. If the analysis of photography was performed completely, then this shortcoming would be eliminated. For the purposes of this example, this is unessential.

Figure 2 and 3 provide a graphic representation of the data presented in Tables 1 and 2, respectively. Thus, Figure 2 provides a graphic representation of the "used in the formulation of" relation. A directed line leading from one box in Figure 2 to another indicates that the Term represented in the first box is used in the formulation of the Statement represented in the second box. In Figure 2 (and in Figure 3) a dot on the intersection of two lines indicates that they are connected. The lack of a dot indicates the lines merely cross and are not connected.

To interpret Figure 2, consider for example PT1. Following the line from the bottom of the PT1 box leads to a two-way connection. Following the right branch, PT1, is used in the formulation of R8, R11, and R12. Returning to the junction and following the branch to the bottom of the figure, it shows that PT1 is used in the formulation of D1, D2, D6, D9 and D11. Figure 2 can also be used to determine which Terms are used in the formulation of a given Statement. Consider, for example, R11. The arrow to the left of the R11 box leads back to PT1 and PT6. The arrow to the right of the R11 box leads to ST3, ST5, and ST22. That these are the only Terms used to formulate R11 can be verified by the entries in Table 1.

The interpretation of Figure 3 is entirely similar to the interpretation of Figure 2. For example, consider the following question: The Definition of Secondary Term 20, D20, is used to prove which Results?" Figure 3 shows that D20 is used in the proof of R8, R9, and R10.

71

Figure 2 – Graphic Representation of the "Used in the Formulation of" Relation for the Photography Representation

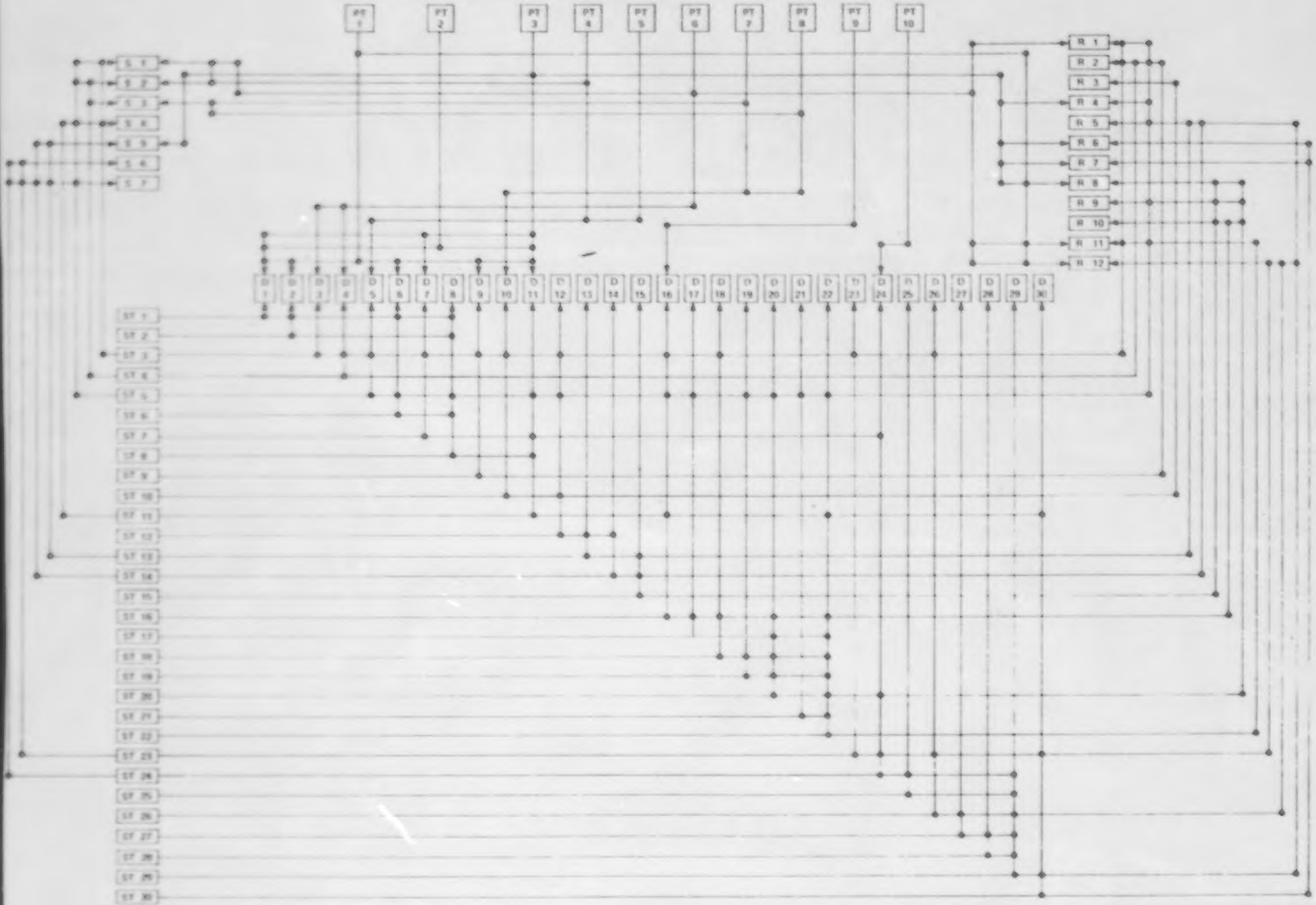
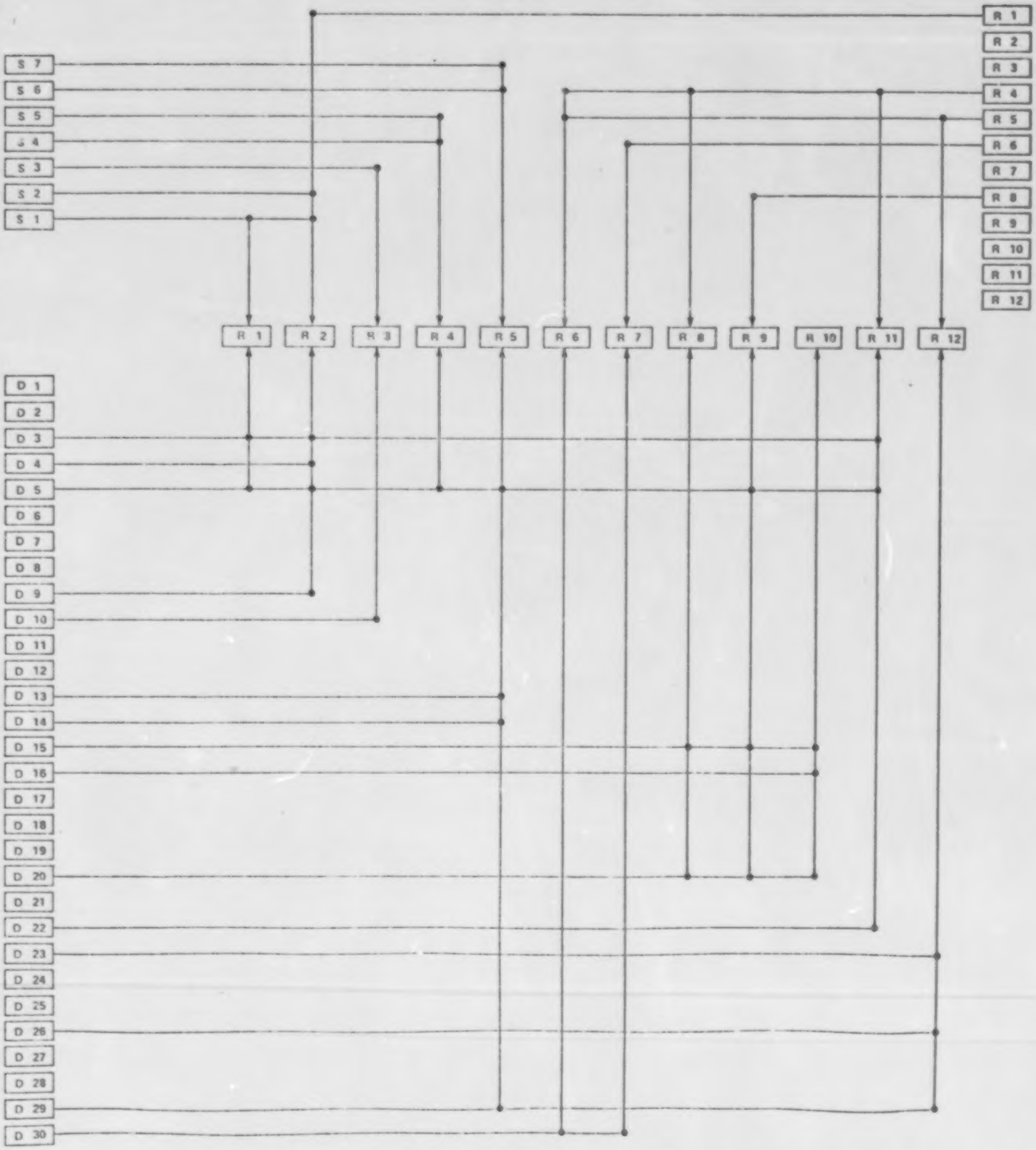

PT 1
PT 2
PT 3
PT 4
PT 5
PT 6
PT 7
PT 8
PT 9
PT 10
S 1
S 2
S 3
S 4
S 5
S 6
S 7
R 1
R 2
R 3
R 4
R 5
R 6
R 7
R 8
R 9
R 10
R 11
R 12
D 1
D 2
D 3
D 4
D 5
D 6
D 7
D 8
D 9
D 10
D 11
D 12
D 13
D 14
D 15
D 16
D 17
D 18
D 19
D 20
D 21
D 22
D 23
D 24
D 25
D 26
D 27
D 28
D 29
D 30
ST 1
ST 2
ST 3
ST 4
ST 5
ST 6
ST 7
ST 8
ST 9
ST 10
ST 11
ST 12
ST 13
ST 14
ST 15
ST 16
ST 17
ST 18
ST 19
ST 20
ST 21
ST 22
ST 23
ST 24
ST 25
ST 26
ST 27
ST 28
ST 29
ST 30

Figure 3 – Graphic Representation of the "Used to Prove" Relation for the Photography Representation


R 1
R 2
R 3
R 4
R 5
R 6
R 7
R 8
R 9
R 10
R 11
R 12
S 7
S 6
S 5
S 4
S 3
S 2
S 1
D 1
D 2
D 3
D 4
D 5
D 6
D 7
D 8
D 9
D 10
D 11
D 12
D 13
D 14
D 15
D 16
D 17
D 18
D 19
D 20
D 21
D 22
D 23
D 24
D 25
D 26
D 27
D 28
D 29
D 30

Finally, we simply note that the set of Possible Statements in this Representation is infinite due to the inclusion of English in L. Also the "can be used to prove relationship", $\models$, need not be of concern to us herein.

This completes the construction of the example. We now prove that the example is indeed an Axiomatically Structured Representation.

Consider Assumption 1 with respect to this example. Every Term (PT and ST) is a recognizable expression in English augmented by elementary Physics and Chemistry. Thus, every Term appears in the lexicon of L and as such no new transformation rules need to be added to the grammar for L. Also these Terms are used in the expected way so that transformation rules for L need not be altered. Whence, Assumption 1 is satisfied by the example.

The set of 49 Statements clearly satisfies Assumption 2. PT has 10 members which are distinct from the 30 members in ST, so Assumptions 3 and 4 are satisfied. Every element in ST is associated with a unique definition rendering Assumption 5 true in the example. Assumption 6 is satisfied since every Secondary Term is used to formulate its own Definition. Furthermore, the list of results does not include any Definitions. Thus, Definitions are not proved and Assumption 7 is seen to be satisfied. Assumption 8 is also obviously true.

Assumption 9 is concerned exclusively with Secondary Terms. Definitions 1 through 30 can be seen to be equivalences in which the Defined Secondary Term is asserted to be equivalent to a phrase containing only Primary Terms and Secondary Terms. Thus, whenever a Secondary Term appears in a Possible Statement it is a straightforward process to replace the Term with its equivalent defining phrase.

For example, consider the following Possible Statement.

s: Lenses break.

We must find a new Possible Statement s_1 that satisfies Assumption 9. That is, we must eliminate the Term lens in s. The defining equivalence for "lens" from D1 can be used to construct s_1.

s_1: Devices for collecting and focusing light rays reflected from the subject break.

Notice that "lens" does not appear in s_1. In fact, all Terms in s_1 either appear also in s ("break") or else they appear in D1 ("devices", "collecting", "focusing", "light rays","reflected" and "subject"). It also is clear that D1 can be used to prove that s and s_1 are equivalent. That is,

$$\{D1\} \models s \leftrightarrow s_1$$

holds. Whence, assumption 9 holds for our example.

It should be clear that assumption 10 will hold in the example. That is, equivalences such as our Definitions do not strengthen an Axiomatically Structured Representation in the manner expressed by assumption 10. Assumptions 11 and 12 can be seen also to hold for this example.

Therefore, we have constructed an example of an Axiomatically Structured Representation of the subject matter we have called photography.

This example permits us to make an interesting point concerning the Reduction of Axiomatic Representations to Primitively Based Representations. Consider D8, the Definition of "camera" in terms of "lens", "diaphragm", "film", and "shutter". Suppose we apply D6 to eliminate "shutter" from D8 in accordance with assumption 9. We have then, after rearranging the resulting Statement:

$D8_1$: A camera is a device consisting of lens, diaphragm, film holder, and a device that controls the length of time light from the lens is allowed to strike the film.

Suppose we apply assumption 9 again this time to $D8_1$, and continue this way with our goal being to construct $D8_n$ in which only Primitive Terms appear. Without going into the details we can see that $D8_n$ will be a Statement containing on the order of 100 words. Secondary Terms then serve the necessary function of breaking complex notions down into units of L that are understandable and manageable.

Appendix II presents the matrices that result from applying the above described matrix method to this example. In Appendix II Figures 1 and 2 are the Adjacency Matrices for Syntactic and Deductive Dependency, A(SD) and A(DD), respectively. Figure 3 is the Adjacency Matrix for Dependence: $A(DD)=[A(SD)+A(DD)]\#$. In this example it was found that $A(DD)^{16}=[0]$ and, thus, the maximal length dependency chain is 15. That is, $m=15$. Figure 18 is the complete Dependency Matrix D for Photography:

$$D = \left[\sum_{i=1,15} A(D)^i\right]\#$$

As an example of the use of these matrices consider Figure 17 showing a 15 link Dependency sequence from PT6 to R7. For illustrative purposes we have identified this sequence:

R7 Depends on R6
R6 Depends on D30
D30 Depends on D29
D29 Depends on D25
D25 Depends on D24
D24 Depends on D20
D20 Depends on D19
D19 Depends on D18
D18 Depends on D16
D16 Depends on D11

D11 Depends on D8

D8 Depends on D6

D6 Depends on D5

D5 Depends on D3

D3 Depends on PT6.

Our final task, to which we now turn, is the development of prescriptions to act as guidelines in developing Axiomatically Structured Representations of subject matters. We need guidelines that will guarantee that Assumptions 1 through 12 will be satisfied. Whence, it is natural, as we have done, to develop these prescriptions directly from the Assumptions. We list these prescriptions with minimal comments.

We caution the reader not to expect from the prescriptions a step-by-step process that will result in an Axiomatically Structured Representation for any subject matter. Such a template would be impossible to formulate. As we pointed out above, the instructor has numerous degress of freedom that can be exercised while structuring any subject matter. The instructor, at least, is free to:

(1) choose any underlying language

(2) select any set of Primitive Concepts

(3) select any set of Primary Terms to represent Primitive Concepts

(4) select any set of Defined Concepts

(5) select any set of Secondary Terms to represent Defined Concepts

(6) select any set of Statements

(7) select any Statements as Definitions

(8) select any Statements as Suppositions

(9) select any Statements as Results

(10) formulate any Statements as deemed appropriate

(11) prove any Statement as deemed appropriate.

The selection, formulation and proof policies pursued by the instructor are constrained only by assumptions 1 through 12.

Whence, it should be clear that there is not a unique subject matter Representation for a given subject matter. This situation presents no problems and, in fact, seems to be desirable. Individual instructors comprehend the same subject matter differently. Given the opportunity they will develop diverse structuring of subject matter elements based on their diverging comprehension. Rather than inhibit the individual orderliness they impose on the subject matter, their unique contributions should be encouraged.

From this point of view we seek minimal prescriptions that facilitate rather than inhibit creative subject matter structuring. The prescriptions simply provide the minimal ground rules to be followed in order to ensure that the subject matter structuring will exhibit interal consistency, and will be suitable as a basis for Information Transfer instruction.

* Select and use as natural a language as possible (e.g., English).

* Be prepared to provide necessary assistance in comprehending the selected language. Both syntactic and semantic (definitional) assistance should be provided.

* Clearly identify all auxiliary knowledge the learner is assumed to possess, as any specialized knowledge must be considered as contributing to L, the underlying language.

- Be prepared to provide necessary assistance in the comprehension of any required specialized knowledge.
- Select as Primary Notions concepts that are readily comprehensible and possess inherent intuitive appeal.
- Select as Primary Terms to represent Primary Notions terms that are natural to use and that will not result in semantic confusion.
- Select as Secondary Notions concepts that are both meaningful and useful to the learners.
- Select as Secondary Terms to represent Secondary Notions terms that are natural to use. Do not introduce new Secondary Terms (e.g., new words) unless elements in L cannot be found to serve the purpose of the Defined notion.
- If new terminology is adopted be certain to provide information concerning the syntax of such terminology.
- When terms in L are used to define Secondary Notions, carefully evaluate the syntactic use of these terms to ascertain whether or not old syntactic uses satisfy the new syntactic functions. Describe carefully all syntactic deviations that are uncovered.
- At least one, but at most finitely, many Primary Notions and thus Primary Terms must be selected.
- If Secondary Notions are to be Defined, select the corresponding Secondary Terms in such a way that the Secondary and Primary Terms are clearly distinct and distinguishable.
- Every Secondary Notion must be Defined, and, thus, every Secondary Term must appear in at least one Sentence: viz. its Definition.

- Label subject matter related Statements clearly and distinctly.
- Distinct Secondary Terms must have distinct Definitions. That is, do not define more than one Secondary Term (Notion) at a time.
- Use only equivalences and equalities when defining Secondary Terms and Notions. (This ploy helps to assure that Assumption 9 will hold.)
- In selecting Suppositions select statements that possess strong intuitive appeal.
- In selecting Results select only results that aid in the understanding of the subject matter.
- Verify that claimed Results can in fact be proved and be prepared to exhibit those proofs on demand.
- Avoid cyclic definitional sequences. (See assumption 11.)
- Avoid cyclic derivational sequences. (See assumption 12 and theorem 7.)

REFERENCES

Stelzer, J. and Kingsley, E. An Axiomatic Theory of Subject Matter Structure, HumRRO Technical Report 74-14, Alexandria, VA, 1974(a).

Stelzer, J. and Kingsley, E. "Axiomatics as a Paradigm for Structuring Subject Matter," Instructional Science, December, 1974(b).

Suppes, P. Introduction to Logic, Van Nostrand Reinhold Company, Princeton, NJ. 1957.

Chapter II

CLASSIFYING SUBJECT MATTER CONTENT

In this chapter we will discuss the problem of classifying instructional subject matter. We are specifically interested in classifying subject matter on the basis of the content of the subject matter. That is, we want to identify classifications on the basis of attributes exhibited by the subject matter content. Therefore, purely extrinsic characteristics such as the apparent difficulty people have learning the subject matter, the usefulness of the subject matter, the applications of the subject matter, etc., will not contribute to our classificatory schema.

However, some properties which may *seem* to be extrinsic will be used in classifying subject matter content. In particular, are those extrinsic attributes that arise due to intrinsic structural properties? For example, consider the property of one subject matter being an analogy for another. Without defining this concept precisely at this point (the precise definition is provided below), it should be clear that two subject matters will be analogous just in case they have similar structural properties. In a sense, analogy is then an extrinsic attribute arising from intrinsic structural characteristics.

Our interest in subject matter classification can be stated simply. Confronted with the problem of providing instruction in a particular subject matter, we seek guidance in making the instructional process as effective as possible. We expect the classificatory scheme to contribute to this goal. For example, suppose we have at hand an instructional process that is reasonably effective and efficient for some particular subject matter. Suppose further that we have classified this subject matter's content with another subject matter for which instruction has not been, but will be, provided. Hopefully, our classification will help to effect a transfer of methodological principles to the instructional process for the new subject matter that will facilitate the development of effective, efficient instructional processes for this subject matter.

We emphasize that we do not aspire to find classifications that will guarantee that methodological principles will transfer profitably between subject matters. Such an aspiration certainly would be unrealistic. There is simply no valid reason why methodological principles effective in one domain will be effective necessarily in another, even if the subject matters are classified together. Accordingly, as stated above, our goal is the more modest goal of seeking classifications that will increase the probability of positive transference of methodological principles.

A classificatory scheme also has another possible and important application. It may be possible to transfer knowledge of one subject matter to knowledge of another subject matter. Knowing about optics may be useful in developing a knowledge of photography, for example. This is the potential usefulness of a concept such as analogy. A classificatory schema should also assist us in identifying analogical relationships that may be used to facilitate transference of preexisting learner knowledge.

Expressing in this way the value of any content classification scheme helps us to understand more clearly the form our classification will take. In particular, we do not seek classification of subject matter content into mutually exclusive categories. Given a new subject matter, we want to survey the landscape, so to speak, eliciting help from whatever quarter it can be drawn. If we can find several meaningful categories for the subject matter, so much the better, since we will then have available a larger pool of methodological principles and analogies from which to draw assistance. Of course, the classification scheme must be restrictive enough so that some distinctions are made or else the classification would simply be meaningless.

Whence, we seek a classification scheme for subject matter content to assist us in choosing methodological principles for instruction, and to assist us in transferring comprehension. The schema will not necessarily result in the classification of subject matter into mutually exclusive categories. Furthermore, the classification will be based on intrinsic content oriented characteristics, including structural, rather than extrinsic non-content characteristics.

In order to discuss content characteristics, we need some mechanism within which content can be represented. That is, to compare, contrast and classify content we must assume that the various subject matter contents are organized in some standard fashion. Without such an organization, it is difficult to see how to compare content elements. Accordingly, we will base our classification scheme on properties

84

identifiable within Axiomatically Structured subject matter Representations (Defined in Chapter I, p. 25). This means that the first step in applying the classification scheme developed herein to an instructional subject matter is to develop an Axiomatically Structured Representation, or simply Representation, for the subject matter. The characteristics of this Representation will be capable, in turn, of being utilized to classify the subject matter content.

Our classification development will begin with a discussion of surface features exhibited by a Representation. We will next consider deeper structural characteristics exhibited by Representations. Finally, we will consider extrinsic characteristics that are based on inherent properties of Representations, i.e., structure. In particular, we will discuss the concept of an analogy for a subject matter. We stress, however, that we will discuss all properties of Representations that may be of use in classifying content.

The first, and most crude, level of classification arises from a simple statement of the general area in which a subject matter falls. As examples, "electronics," "transistor theory," "physics," "optics," "photography," "Ohm's Law," "parallel circuits," "film emulsions," "probability," "decision theory" all serve to classify subject matter at a gross level. While not providing much useful specific information, such a classification begins to establish a context in which to interpret deeper levels of classification.

Surface classifications arise from the particular attributes of the Representation (Chapter I) of a Subject Matter. The first classificatory scheme of this type is the Representation's underlying language: L. That is, we can specify for each Representation the underlying language: English, Russian, French, etc. This classification is non-trivial since a specification of L includes a precise statement of those areas of knowledge subsumed within L. For example, a Representation of photography may be developed within English (L) that subsumes the terminology of optics as part of L. The Representation of Photography in Chapter I can be seen to have this characteristic in that optical terminology is freely employed as part of L.

Thus, a classification of Representations based on L begins to exhibit internal content similarities. Beyond the natural language comparisons, we can compare and contrast Representations based on subsumed knowledge. It is of interest, for example, to discover that two distinct subject matters have Representations both in English and both subsuming a knowledge of elementary optics. One of the values of developing precise Representations of subject matters is that it forces one to recognize explicitly the complete knowledge space upon which the Representation rests. Implicit assumptions of this type when brought to the surface often can result in modifications of the instructional process. Comparisons of Representations on characteristics once implicit, but now explicit, may lead to new groupings of subject matters once thought to be inappropriate.

Another Surface classification arises from a consideration of the Primary Notions on which a Representation is based. As with subsumed knowledge, rarely, if ever, do we encounter statements such as: "We will base our understanding of subject matter x on the following primitive concepts." Such clarity in instructional processes would seemingly be of major classificatory importance. For example, it may be that the Primitive Notions for one subject matter Representation are wholly, or even partially, contained within the set of Primitive Notions for another. In such a case, the first subject matter may be profitably thought of as a conceptual subpart of the second subject matter. In the case of partial containment, we would see that the first subject matter is partially a subpart of the second. Comparisons of this type are not usually forthcoming, but naturally arise if we have available an Axiomatically Structured Representation for each subject matter.

In classifying content in terms of Primary Notions, it is important to maintain the distinction between the Notion and the Term (a member of PT) used to represent the Notion. That is, we do not classify content on the basis of the actual Primary Terms employed in the Representation. The Terms are only linguistic devices. We must look beyond the particular Term to the actual Notion being represented. Thus, the classification is based, appropriately, on the Primary Notions rather than the Primary Terms.

Another Surface classification is the set of Secondary Notions appearing in a Representation. As with Primary Notions, one subject matter Representation may employ Secondary Notions that are contained in, or overlap, with another subject matter's Secondary Notions. However, since Secondary Notions are entirely eliminable (Theorem 2, Chapter I), a classification based on Secondary Notions is not as important as a classification based on Primary Notions. That is, the Primary Notions represent an integral characteristic of a given Representation. They provide the conceptual "cornerstone" for the entire subject matter. Secondary Notions do not play such a pivotal Representational role since they are optional and eliminable. However, it is of interest to compare and contrast subject matters on the basis of Defined or Secondary Notions.

As with Primary Notions, to classify Representations on the basis of Secondary Notions, we must look beyond the Secondary Terms (ST) to the Notions actually Represented. That is, it is of minimal interest to discover that two subject matters have similar linguistic entities: Terms. But it is most interesting to discover that two subject matters contain the same conceptual frameworks (Notions) no matter what Terms are used to represent these Notions.

Whence, Surface classifications are linguistic in that the specification of L is a linguistic classification. Surface classifications are conceptual in that they specify the subsumed conceptual framework within L, as well as the Primary and Secondary Notions within the Representation.

Recall from Chapter I that the set of Statements (S) within a Representation consists of three types of Statements: Definitions, Suppositions and Results. It may be felt that these categories themselves may be used to provide another set of meaningful classificatory categories. However, this is not the case. Definitional categories are provided for when Representations are classified by Secondary Notions.

Suppositions in Representations play the role of axioms in axiomatics. It is extremely unlikely that diverse subject matters will be based, respectively, on axioms that have equivalent meanings. (As in classifications based on Notions, when considering Suppositions we must look beyond the syntactic form of a Supposition to its intended meaning). In other words, if two Representations are based on respective Suppositions having the same meanings, then they will *not* be diverse, i.e., they will be classified together obviously. Thus, classifications based on Suppositions in Representations will not provide new information.

Similarly, classifications based on Results (theorems) in a Representation will not lead to new information. If Results having the same meanings are included in two Representations, then obviously the Represented subject matters are similar.

At this point, we have discussed four of the eight surface characteristics exhibited by Axiomatically Structured Representations. The remaining four characteristics do not contribute to meaningful Surface classifications. The set of Possible Statements (PS) arises from the Underlying Language (L) in conjunction with the Primary and Secondary Terms (PT and ST). Thus, a classification based on PS cannot be expected to contribute new information.

Consider the "used to formulate" relationship (F). This relationship is employed within the Extended Underlying Language (L^1) and also provides no new information. The "used to prove" relationship ($\vdash$) can only contribute classificatory information concerning how proofs are conducted. For example, a classification based on logical practices that are employed in proofs is possible. But these conventions such as Sentential Logic, Quantificational Logic, etc., arise from subsumed knowledge within L. This information contributes to the already described conceptual classification associated with L. Thus, the "used to prove" relationship does not provide additional classificatory information. Similar remarks apply to the "can be used to Prove" relationship ($\models$).

Thus, Surface classifications providing meaningful classificatory information are based only on the Underlying Language, the Primary Notions and the Secondary Notions. The other five components of any Representation (PS, S, F, $\vdash$, $\models$) either provide only trivial information, or provide information that is redundant given the Surface classifications based on the above three mentioned characteristics.

The Surface classifications described above are based on apparent properties. For example, the classification based on Primary Notions relies on the intended meaning of the Notion. Such a classification ignores structural properties of Notions. We may desire to classify content on the basis of Notions that have similar logical properties. For example, two subject matters may each be based on Primary Notions that are only reflexive, symmetric and transitive binary relations. Structural similarities of these types may lead to useful classifications. Similarly, Suppositions for various Representations may not have the same meaning but may exhibit the exact same structural properties. In such a case, we may want to classify these Representations together. A detailed investigation of these structural properties, as opposed to Surface properties will be undertaken below. Indeed, we will define and investigate the notion of analogy and relate the concepts we introduce to the problem of classifying content.

Before turning to this task, we will discuss classificatory properties that are identifiable from the concepts of Dependence as defined for Representations.

The first structural classification with which we will be concerned is a measure of the Linearity of the content. Recall from Chapter I that m represents the maximal length dependency chain for an Axiomatically Structured subject matter Representation. The content Linearity will be defined by a comparison between m and the size of the set of Statements, S, in the Representation. As a preliminary definition in this development we assign a specific name to the size of S.

Definition 1. Let R be an Axiomatically Structured Representation for a subject matter and let S be the set of Statements in R. The number of elements in S is represented by n_R.

In set theoretic terms, we can say that n_R is the cardinality of the set S. When there is only one Representation under consideration, we will write n in lieu of n_R. Thus, n is the cardinality of S for a particular Representation R.

Consider a very simple example of a completely linear set of Statements, s_1, s_2, s_3.

s_1 is the Definition of a Secondary Term t.

s_2 is a Supposition formulated with t.

s_3 is a Result whose proof employs s_2.

In the formal notions of Chapter I this means we have both

$$Fts_2,$$

and a set A such that $s_2 \epsilon A$ and

$$A \vdash s_3.$$

From Definition 4 we have

s_2 is Syntactically Dependent on t,

and from Definition 5 we have

s_3 is Deductively Dependent on s_2.

Applying Definition 6 yields

s_3 is Dependent on t.

In this case, the maximal length dependency chain, m_R (or m), for the Representation is 2 and, thus, m=2 while n=3.

We desire our linearity measure to be absolute. This means that if two Representations are assigned the same numerical measure of linearity, then they will exhibit identical relationships between their respective m and n, independent of the actual magnitude of these elements. Thus, it is appropriate to consider the following ratio as a preliminary measure of content linearity.

$$\frac{m}{n}$$

However, it would be pleasant to restrict linearity measures to the interval 0, 1 which means we must use n-1 rather than n. Thus, linearity becomes the ratio:

$$\frac{m}{n-1} .$$

In the above completely linear Representation, we have for linearity

$$\frac{m}{n-1} = \frac{2}{3-1} = 1.$$

Notice also that if there are no dependency relationships, then m=0, and whence the above formula results in a linearity measure of 0.

However, in general, it is possible for Primary Terms to contribute to m. As an example suppose we expand the above Representation to include 3 Primary Terms t_1, t_2, t_3. Assume also in this case that

s_1, the Definition of t is formulated in
terms of t_1

and that

s_2, s_3 are as before.

We now have

$$Ft_1t^*,$$

$$Fts_2\ ,$$

and

$$A \vdash s_3$$

where $s_2 \epsilon A$. It follows that m=3. Now applying the above linearity measure yields 3/2 which does not satisfy our restriction to the interval 0, 1.

The problem that arises here is that the inclusion of Primary Terms allows m to increase by one. But notice that m can increase at most by 1, since Primary Terms may not be interrelated by Dependency. That is, our second example above represents a specific example of the most general possible completely linear Representation. To have the linearity measurement of 1 in this case we must increase the denominator in our preliminary formula by 1 to account for the possible unit increase in m. That is, the denominator becomes n-1+1=n.

This discussion serves to justify the following definition.

Definition 2. The linearity, ℓ_R, for an Axiomatically Structured Representation, R, is defined as follows.

$$\ell_R = \frac{m_R}{n_R} .$$

In the degenerate case when n=0, we set $\ell_R=0$.

When no confusion is possible concerning which Representation is being referenced, we will write ℓ in lieu of ℓ_R. That is,

$$\ell = \frac{m}{n} .$$

The inclusion of the degenerate case in Definition 2 is necessitated by the fact that n may be 0. That is, we may encounter a Representation that includes no Statements: in which the set S is the empty set. In such a case nothing is asserted about the subject matter. In practice, we do not expect to encounter degenerate Representations of this type, but for purely formal reasons we must provide for their occurrence.

We can prove formally that $0 \leq \ell \leq 1$. In the degenerate case, n=0 and by Definition 2, $\ell=0$. In the non-degenerate case, $n \neq 0$, and, in fact, $n>0$. The maximum value for m, excluding elements in PT, is n-1. At most, one element in PT can contribute to m. Thus, the absolute maximum possibility for m is n. Whence, we see that $\ell \leq 1$. Furthermore, $m \geq 0$ in the non-generate case establishing our result.

Linearity provides a useful classification for content Representations. Assume that instructional sequencing is constrained so that all elements that an element, s_1, depends upon are introduced or explained prior to instruction on s_1. Thus, all Secondary Terms s_1 Depends on are Defined and explained prior to instruction on s_1. Also instruction on s_1 is preceded by instruction on all Statements, s, such that s_1 is Deductively Dependent on s. Primary Terms need not be considered since they are assumed to be intuitively comprehensible. That is, strictly speaking, instruction on the Primary Terms need not be considered as part of the instructional sequencing. We might expect any possible confusion concerning Primary Terms to be cleared up at the outset of instruction. Following this process, we must next concern ourselves with sequencing instructional components.

Within such an instructional setting consider a Representation for which $\ell=1$. It follows that there is only one possible instructional sequence. In fact elements must be introduced and explained in the exact linear sequence determined by the Dependency relationships. As a more concrete example, consider the first Representation example above where PT is empty and $S = \{s_1, s_2, s_3\}$. In this case we must sequence instruction as follows.

Step 1. Define and explain t.

Step 2. State and explain s_1.

Step 3. State, explain and prove s_2.

Within any of these steps indicated operations may be omitted, amended or resequenced. That is, Step 3 may simply be: "State s_3." Step 3 might also be: "Make a Statement equivalent to s_3." The allowable variations within steps are beyond the scope of this Chapter and will not be dealt with herein.

At the other end of the spectrum, where $\ell=0$ and $n>0$, there are no interdependencies among the Representation elements. In this case total freedom is possible in sequencing. Since there are no Dependencies, elements can be Defined, introduced, proved, explained, etc., in any sequence.

It is not possible to discuss, a priori and meaningfully, in-between points of linearity. That is, it is not clear exactly what a linearity of 1/2 implies. It is not even clear that Representations should be classifed by single values rather than intervals. For example, it may be more meaningful to classify content by four (arbitrarily chosen) intervals such as the following.

(1) $0 \leq \ell < 1/5$

(2) $1/5 \leq \ell < 1/2$

(3) $1/2 \leq \ell < 3/4$

(4) $3/4 \leq \ell \leq 1$.

Meaningful distinctions of this type for classificatory purposes must await empirical results. We leave as an open question, awaiting empirical evidence, the problem of identifying meaningful content classifications based on Linearity.

The second structural classification concerns the (inter)Relatedness of Representation elements. As with Linearity we seek a numerical parameter to characterize this notion. The idea behind Relatedness is quite simple. Consider the complete Dependenty matrix, D_R (or D), for a Representation R. Assume that elements of R appear in the rows and columns of D in such a way that elements of PT are first followed by elements of ST (Definitions) followed by Suppositions followed by Results. The matrix D can only have non-zero entries, or ones, in that area of the matrix lying above the main diagonal, and to the right of the rightmost column representing a Primary Term. The Relatedness of R will be the ratio of ones occurring within this area to the total number of possible entries in this area of the matrix.

To make these notions more precise let us consider a specific example. Let R be a Representation in which there are three Primary Terms and four Statements (n=4). The complete Dependency matrix has the following structure where an X indicates an entry of either a zero or a one.

$$\begin{array}{c|ccc|cccc}
 & \multicolumn{3}{c|}{\overbrace{\qquad\qquad}^{PT}} & \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad}^{S}} \\
 & X & X & X & X & X & X & X \\
PT & X & X & X & X & X & X & X \\
 & X & X & X & X & X & X & X \\
\hline
 & X & X & X & X & X & X & X \\
 & X & X & X & X & X & X & X \\
S & X & X & X & X & X & X & X \\
 & X & X & X & X & X & X & X
\end{array}$$

As stated above, we can show that the only area in this diagram that can have non-zero entries lies above the main diagonal and to the right of the PT area. Diagrammatically this area can be indicated by replacing X's within this area by 0's.

$$\begin{array}{c|ccccccc} & & PT & & & S & & \\ \hline & X & X & X & 0 & 0 & 0 & 0 \\ PT & X & X & X & 0 & 0 & 0 & 0 \\ & X & X & X & 0 & 0 & 0 & 0 \\ & X & X & X & X & 0 & 0 & 0 \\ S & X & X & X & X & X & 0 & 0 \\ & X & X & X & X & X & X & 0 \\ & X & X & X & X & X & X & X \end{array}$$

We need a general process to count these entries of 0. The following expression performs this task.

$$q_R = \sum_{i=p_R,p_R+(n_R-1)} i \quad .$$

where

p_R = the number of elements in PT

n_R = the number of elements in S
(Definition 1 above).

In the above example, we have

$p=3,$

$n=4,$

and, thus

$$\sum_{i=3,6} i = 3+4+5+6 = 18$$

as appropriate. In addition, let

$$e_R = \text{the number of non-zero entries in } D_R.$$

The formal definition of Relatedness is as follows.

Definition 3. The Relatedness for an Axiomatically Structured Representation, R, is defined as

$$r_R = \frac{e_R}{q_R}$$

When there is no possibility of confusion, we can drop the subscript R; yielding for r the following expression:

$$r = \frac{e}{q}$$

$$= \frac{\text{the number of non-zero entries in } D}{\sum\limits_{i=p,p+(n-1)} i}$$

Unlike Definition 2, we have chosen not to include in Definition 3 a statement of the degenerate case. There is one possible degenerate case when the denominator of the expression defining r becomes 0, rendering r undefined. If n=0, then

$$\sum_{i=p,p+(n-1)} i = 0,$$

and we let r=0 by definition. This is reasonable since n=0 means there are no Statements and, hence, no Dependencies. Thus, there is no Relatedness, and, appropriately, we have r=0.

Therefore, it should be clear that, in general,

$$0 \leq r \leq 1$$

and larger values of r indicate a larger degree of (inter)Relatedness among Representation elements.

An interesting question arises concerning the relationship of r to ℓ. In general, it can be asserted that $e \geq m$. Also, if $m \geq 1$, then $e \geq m+1$. This later statement follows since if m is the maximal length dependency chain, then there are at least m+1 elements Dependently interrelated by the maximal chain. Thus, there are at least m+1 entries in D.

It may be possible to relate more precisely r and ℓ through the introduction of additional R-related parameters such as m, n, e, q. However, the benefit to be derived from such distinctions is questionable. Thus, we will not pursue this point herein. We leave as a second open problem the value of pursuing these relationships. Perhaps, subsequent empirical research will indicate that these investigations warrant consideration.

Another interesting question concerns the usefulness of r in sequencing instructional units. It seems reasonable to suppose that it may be profitable to treat highly (inter)Related Representations differently than Representations exhibiting low levels of (inter)Relatedness. For example, consider Representations exhibiting a high degree of Relatedness. It may be desirable to provide instruction on elements in such a Representation more "in parallel" than "sequentially." That is, the instructional process might constantly increase comprehension of several elements simultaneously, switching back and forth between them in order to stress their Relatedness. On the other hand, a low level of Relatedness indicates that Representation elements are quite distinct and can be taught more in isolation from one another.

Thus, we suggest that it may be profitable to classify content by the degree of Relatedness exhibited by Representations. The form of such a classification may be based on intervals, as discussed above for Linearity, or may be based on individual Relatedness measures. The most useful form of such a classification must, as with Linearity, await empirical evidence. As acknowledged above, the present study is necessarily *a priori* in nature which means that these questions must be left as open problems.

Before turning to the question of extrinsic classification categories (analogy) based on intrinsic structure, we would like to briefly mention other interesting structural properties. These properties are of interest instructionally it seems, but do not bear directly on the question of content classification. Within a Representation, the most *Salient* concept (or concepts) may be defined as those concepts represented by Terms upon which most other elements Depend. For example, Figure 18 in Appendix 2 is the complete Dependency matrix for the Chapter I Photography example. In this matrix, PT6 is the most Depended upon Term. Thus, the Concept of light, represented by PT6, is the most Salient Concept and seems pivotal in comprehending photography. The concept of emulsion, represented by ST3, also can be seen to be of great importance. It would seem reasonable to provide extra precautions to insure student comprehension of such Salient concepts; for it is difficult to imagine comprehension of photography without comprehension of these Concepts.

Similar to the above notion of Relatedness, we may identify groupings of Representation elements that seem to be highly interrelated. That is, we may identify clusters of elements within a Representation that are highly interrelated. Instructionally, it may be desirable to deal with such clusters more as units. Sequencing may be based on one level on the notion of clusters and on another, lower level on the notion of components internal to clusters. This idea is probably similar to the concept of modularized courses. But with a Representation with a Dependency matrix at hand, it may be possible to provide a more "objective" approach to the problem of modularization. Mathematical techniques using relatedness measures may be employed to define preliminary modularization groupings. Subsequent authoring work may refine, adjust, etc., preliminary groupings based on the author's subject matter expertise.

Pursuance of these parenthetical comments lies outside the domain of this chapter. We now return to our main developments.

Our next, and final, classification category is based on the concept of an analogy. In the most general terms, an analogy is a relation of "likeness" between two things. More specifically, we are interested in characterizing the concept of one subject matter being analogous to another; or one subject matter being an analogy for another. The notion of an Axiomatically Structured Representation provides us with a framework within which this concept can be defined very precisely. Briefly, one subject matter can serve as an analogy for another if two conditions are satisfied. First, each Primary Notion in one subject matter (Representation) must be associated with a Primary Notion in the other subject matter (Representation) in such a way that associated Primary

Notions exhibit the same logical structure: both constants, both one-place predicates, both two-place predicates, etc. Second, the Suppositions (axioms) from one subject matter correspond to Suppositions or Results (theorems) in the other subject matter when Primary Notions in the first subject matter Suppositions are replaced by their associated Primary Notions in the second subject matter. As will be seen, these two conditions, as stated here, only approximate our formal conditions to be introduced. For our purposes, the form of these two conditions suffices at this point informally, to characterize, though slightly incorrectly, the notion of analogy. Before turning to the details of this development, it will be useful to discuss the value of analogies in instruction and learning.

Suppose that instruction is to be offered in some particular subject matter such as electronics. It is clearly recognized that one way of maximizing the effectiveness of the instructional process is to utilize as much as possible the individual learner's experiential background. For example, suppose at some point in the instructional process we desire to explain Ohm's Law: E=IR where E is Voltage, I is Current and R is Resistance. Suppose further that our learner is an experienced plumber or, at least, comprehends the basic principles of plumbing. How can we utilize, or capitalize upon, this learner's preexisting knowledge?

We might point out to the learner that a simple circuit with a battery and resistor, both suitably grounded, is analogous to having a pond with a pump circulating water through a pipe containing a restriction. The restriction may be a reduction from a larger to a smaller pipe in the same direction as the water flow. We may go on to point out that Voltage is analogous to the potential for water flow as established by the pump: water Pressure. Furthermore, Current is analogous to water flow in the pipe: water Flow. Resistance is analogous to a Restriction empeding water flow. Finally, grounding of the ends of the circuit is analogous to having the pump pump water out of the pond through the pipe back into the pond. This situation is depicted graphically in Figure 1 (p. 37).

Ohm's Law in this analogy becomes: Pressure equals Flow times Restriction. Strictly speaking, Ohm's Law cannot be said to hold in the pond/pump analogy since we have not introduced suitable measures of Pressure, Flow and Restriction. But doing so, even if this is possible, need not concern us here. What is important is that we would expect our experienced plumber to recognize immediately that Pressure is proportional to both Flow and Restriction. That is, the learner can be expected to realize that increasing either Flow or (the amount of the) Restriction results in a concomitant increase in Pressure. Conversely, the learner can be expected to realize that reducing either Flow or Restriction reduces Pressure. Similarly, increasing (decreasing) Pressure results in an increase (decrease) of the combination of Flow and Restriction.

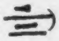
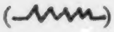
In this sense, an analogy is a transliteration from one subject matter to another. Given such a transliteration the learner can perceive how to apply preexisting knowledge to the new subject matter. It provides the learner with a useful and, from the learner's point of view, natural way to think about an unfamiliar set of concepts. Provided with the precise transliteration associations the learner can even proceed to make "discoveries" about the new subject matter by applying already known principles in transliterated form. For example, in our Ohm's Law/Pump analogy, the learner surely will perceive that breaking the ground connection below the battery (removing the pump feeder from the pond) results in zero Current flow (no water Flow).

Thus, we see that an analogy of this type may lead the learner to discover, or anticipate, important principles in the instructional subject matter. But notice that not all analogous principles "carry back" to the subject matter. For example, suppose we remove from the pond the end of the pipe behind the Restriction; the end returning water to the pond. Suitably connected, the pump will now proceed to pump the pond dry. This result does not apply to the circuit. To break the ground connection behind the resistor, i.e., on the side away from the battery, simply means that Current flow ceases.

This situation does not invalidate the usefulness of the concept of an analogy. Notice that we did not claim that a simple circuit such as we are discussing is an analogy for pond pumping. On the contrary, we set up the pond pumping situation as an analogy for the Ohm's Law instruction. There is no reason to expect the notion of analogy to exhibit symmetry, even though this sometimes may be the case.

Instructionally, identifying the points where the analogy breaks down may even be very useful. Suppose, for example we ask the learner to suggest electronic principles based on the learner's pond pumping knowledge. That is, we ask the learner to propose some electronic hypotheses that suggest themselves from the pond pumping analogy. "Well," the learner might say "if we increase the Resistance enough we'll force the battery to bust the wire." This hypothesis stems, of course, from the perception that if we block the pipe completely yet force the pump to pump, something will give. An incorrect hypothesis such as this, far from being undesirable, provides a wonderful opportunity for the instructional agent to clarify the meaning of an electronic circuit for the learner. By comparing and contrasting circuits and pond pumping in this context, we would expect the learner to develop an accurate comprehension of basic electronic circuits.

In the example we are using, it is expected that the analogy will have limited usefulness. That is, there is a point beyond which the pond pumping analogy ceases to be applicable. The principles of electronics diverge from those of simple pumping to such an extent that the pond pumping analogy must be dropped, eventually, in order to prevent distortions in the learner's comprehension of electronics. In this sense, pond pumping is an incomplete or only an approximate analogy for simple circuits. However, it may be possible to find analogies in other subject areas that are not flawed in this sense. If so, these analogies may be carried forward and continuously employed in order to facilitate learning. Below, we will develop formal mechanisms that permit us to compare and contrast analogies for their relative strengths: both conceptual (syntax) and derivational (principles).

Before turning to these developments, we would like to point out one standard use of analogies in instruction, or explanations in general. We have in mind the concept of an example. What is an example, but an analogy? Suppose in an explanation of Ohm's Law the instructional agent asserts: "As an example of this notion, consider the following circuit" at which point a circuit is drawn, perhaps, on a blackboard. What is the relationship between that circuit and the subject matter being discussed? The circuit is a simple example of an analogy. Analogous to a battery we have a symbol (); analogous to Resistance we have another sumbol (); and so on until all relevant principles and notions find their analogous embodiment.

The value of examples of this nature is that they are, so to speak, the accepted analogy. We agree to recongize such analogies as the standard analogy. Through the standardization we facilitate comprehension and communication. We stabilize the format or analogy in order to minimize accompanying explication thereby permitting us to focus on the new point being explained. Instruction utilizing examples then serves two purposes. On the one hand it facilitates learning by employing preexisting analogies. On the other hand it reinforces the standard analogy thereby laying the groundwork for future reference.

We now turn to the formal characterization of these ideas. In the developments that follow, we assume that

$$R = \langle L, PT, ST, PS, S, F \vdash, \models \rangle$$

and

$$R_1 = \langle L_1, PT_1, ST_1, PS_1, S_1, F, \vdash_1, \models \rangle$$

are both Axiomatically Structured Representations for subject matters. The subject matters for which R and R_1 are Representations mav or may not be the same subject matters. The relationships F and $\models$ are not distinguished for R and R_1 since the context of use will determine whether R or R_1 is being referenced.

We need to be able to compare the logical structure of Primary and Secondary Terms. Any Primary or Secondary Term can be described as having a unique logical structure. A Term may be the name of either an individual or a set. A Term may also be an attribute such as the attribute of redness in which case the Term is a one-place predicate. A Term may also be a relationship which means that it is a binary or two-place predicate. In general, Terms may be n-place predicates where n is any positive integer. If two Terms are both individual names, are both names of sets, or if the two Terms are both n-place predicates, then we say that the two Terms have the same logical structure.

Definition 4. Any two Terms t, t_1 have the same logical structure if and only if either

(i) t, t_1 are both individual names, or

(ii) t, t_1 are both names of sets, or

(iii) t, t_1 are both n-place predicates.

Besides comparing the logical structure of Primary and Secondary Terms, we need to be able to discuss how Primary and Secondary Terms function syntactically. We will say that two Terms t_1, t_2 Function Syntactically the Same (or have the Same Syntactic Functions) if and only if for all well formed Statements s such that Ft_1s, if every occurrence of t_1 in s is replaced by t_2, then the resulting Statement is also well formed. Alternately, we might say that t_1 and t_2 are in the same transformational categories (noun phrase, verb phrase, etc.), or that transformational rules applicable to t_1 apply as well to t_2 and, conversely. We will not formulate a formal definition of this notion as its meaning should be clear.

108

We also must be able to refer to the Suppositions (axioma) and Results (theorems) in various Representations. These sets were informally introduced above in Chapter I (p. 27). We introduce the following notation to provide these theoretical references.

Definition 5. The set of Definitions for an Axiomatically Structured Representation, R, is defined as follows:

$$\mathrm{Def}(R) = \{s \in S : (\exists t \in ST)\, s = t^*\}.$$

Definition 6. The set of Results for an Axiomatically Structured Representation, R, is defined as follows:

$$\mathrm{Res}(R) = \{s \in S : (\exists A \in \mathcal{P}(S))\, A \vdash s\}.$$

Definition 7. The set of Suppositions for an Axiomatically Structured Representation, R, is defined as follows:

$$\mathrm{Sup}(R) = S \sim [\mathrm{Def}(R) \cup \mathrm{Res}(R)].$$

Thus, we see that the set of Definitions for R consist of all Statements used to define Secondary Terms. The set of Results (theorems) for R consist of all Statements proved in R. Any Statement that is not a Definition and that is not proved, is a Supposition (axiom) for R. This is reasonable since any non-definitional statement that asserts a non-proved fact about the subject matter must be assumed to be true.

As an assumed truth such a Statement must be taken to be a Supposition as in Definition 7.

Definitions 5, 6 and 7 correspond to Definitions 15, 16 and 17 in Appendix I, respectively. Thus, as asserted in Theorem 15, Appendix I, Def(R), Res(R) and Sup(R) divide S into three mutually exclusive sets. These three sets are exhaustive in that every element of S falls into one of these three sets.

As a final preliminary definition, we introduce a mechanism through which we can refer to the Primary and Secondary Terms used to formulate Suppositions.

Definition 8. The set of Terms used to formulate Suppositions for an Axiomatically Structured Representation, R, is defined as follows.

$$\text{Supterms}\ (R) = \{t \in PT \cup ST:\ (\exists s \in \text{Sup}(R))Fts\}$$

The necessity for the formal mechanism for Supterms (R) will become apparent when we discuss the definitions of analogy that follow.

As will be seen, an analogy results when certain Suppositions for one subject matter are reinterpreted in another subject matter. To do so means that the Primary and Secondary Terms appearing in the Suppositions selected from one Representation are replaced by Primary and Secondary Terms from the analogous Representation. The replacement of Terms must be done in such a way that every replacement Term has the same logical structure as the Term it replaces. The Statement that results from the replacement of Terms must be comparable to selected Suppositions and Results in the analogous subject matter in specified ways based on the notion of implication.

More precisely, suppose R_1 is intended to be an analogy for R. On the one hand we must be able to describe how much of R's structure carries over to R_1. We will distinguish four possible categories describing four general cases of relative structural characteristics. On the other hand, we must be able to compare the relative strengths of statements analogous to the Suppositions of R. That is, we must be able to describe how strong (derivationally) R's Suppositions are in their analogous form. We will distinguish three possible categories describing relative derivational characteristics. As a result we will have a twelve-fold description of the concept of analogy.

We begin with the general definition of the concept of analogy that provides the first category describing relative structural characteristics.

Definition 9. An Analogy for Representation R is a Representation R_1, where $L=L_1$, and a pair of functions f, g satisfying the following four conditions.

(i) f is a one-to-one function from a non-empty subset, A, of Sup(R) into

$$\mathrm{Sup}(R_1) \cup \mathrm{Res}(R_1).$$

(ii) g is a one-to-one function from

$$B = \{t \in PT \cup ST: \ (\exists s \in A) Fts\}$$

into $PT_1 \cup ST_1$.

(iii) For all t in B, t and g(t) have the same logical Structure and have the same Syntactic Function.

(iv) For all s in A and for all t in B

$$Fts \Leftrightarrow Fg(t)f(s).$$

Condition (i) of Definition 9 asserts that a subset of the set of Suppositions for R is mapped by f into a subset of the union of the Suppositions and Results for R_1. The subset of Sup(R) on which f is defined is assumed to be non-empty to insure that the analogy to R is non-trivial.

111

Thus, f associates some Suppositions for R with unique Statements in R_1 that are known to be true of R_1's subject matter. To say that f is one-to-one means that unique Suppositions in R are associated with unique Statements in R_1. That is, $s_1=s_2$ if and only if $f(s_1)=f(s_2)$. Since f is one-to-one, we can speak of the inverse of f, f^{-1}, which is a one-to-one function from a subset of S_1, the Statements in R_1, into Sup(R).

We can show that the set B in condition (ii) of Definition 9 is a subset of Supterms (R). Thus, g is a mapping from a subset of Supterms (R) into $PT_1 \cup ST_1$. The function g associates each element in this subset of Supterms (R) with a unique element in $PT_1 \cup ST_1$ that has the same logical structure [condition (iii), Definition 9]. Finally, condition (iv) asserts that a Term is used to formulate a Supposition in A just in case the image under g of the Term is used to formulate the image under f of the Supposition.

Condition (iii) also asserts that for all Terms t in B, t and g(t) have the Same Syntactic Functions. One way to make this assumption clearer is to consider L^1 the extension of L and L_1^1 the extension of L_1 (also L). To say that t and g(t) have the Same Syntactic Functions is to assert that transformation rules in L^1 applicable to t have corresponding transformation rules in L_1^1 applicable to g(t). We might say transformation rules in L^1 applicable to t have "analogous" corresponding transformation rules in L_1^1 applicable to g(t). When L^1 and L_1^1 have the same transformation categories, which will usually be the case since $L=L_1$, this simply means that t and g(t) are in the same transformational categories. In the rare instances when L_1 and L_1^1 have divergent transformational categories, we may have to inspect individual transformation rules. But we really do not expect to encounter such cases so that our rather informal treatment of the notions underlying the notion of Same Syntactic Function are sufficient for our purposes.

The notion of analogy can be represented diagrammatically as shown in Figure 2. In Figure 2, R and R_1 are represented by enclosed spaces. The size and shapes of these and other spaces do not represent specific size information or even relative size information. The spaces simply serve to delineate the scope of R and R_1. The set A is represented as a space wholly contained within Sup(R); or A is a subset of Sup(R), and Sup(R) is in turn contained in R. The function f represented by an arrow maps A into some subset of $\mathrm{Sup}(R_1) \cup \mathrm{Res}(R_1)$. Similarly, g maps a subset, B, of Supterms (R) into a subset of $PT_1 \cup ST_1$. Whence, Figure 2 serves to represent structural relationships existing between R and R_1 when R_1 is an analogy for R.

It is important to realize that an Analogy is a triple consisting of a Representation (R_1) and two functions (f and g). That is, one Representation, R_1, can serve as an Analogy for another, R, in various ways depending upon the definitions of f and g. Below we distinguish three types of Analogy. The Representation R_1 with two specific functions f, g cannot be more than one type of Analogy. But by redefining f and g, while keeping R_1 constant, we can generate more than one type of Analogy. Thus, the structure of R_1 alone does not classify uniquely the kind of Analogy. We must also take into consideration the functions f and g when classifying Analogies. This point will become more clear when we discuss specific examples below.

113

It will be useful to be able to refer to the Domain and Range of functions f and g. The Domain of a function f, denoted by $\mathcal{D}(f)$, is the set of elements upon which f is defined. In Definition 9, $\mathcal{D}(f)=A$ and $\mathcal{D}(g)=B$. The Range of a function f, denoted by $\mathcal{R}(f)$ is the set of images, or values, of the function. In Figure 2, $\mathcal{R}(f)$ is the small space within $Sup(R_1) \cup Res(R_1)$ and $\mathcal{R}(g)$ is the small space within $PT_1 \cup ST_1$. Since f and g are both one-to-one, the number of elements in $\mathcal{D}(f)$ equals the number of elements in $\mathcal{R}(f)$; and the number of elements in $\mathcal{D}(g)$ equals the number of elements in $\mathcal{R}(g)$.

Throughout the following discussion "f" and "g" are reserved for and refer only to the two functions referenced in Definition 9, conditions (i) and (ii), respectively.

Our first refinement of the notion of an Analogy is the notion of a Complete Analogy. A Representation, R_1, with two functions f and g provides a Complete Analogy for R if all Suppositions in R are mapped into the Suppositions and Results for R_1.

Definition 10. A Complete Analogy for Representation R is a Representation R_1, and a pair of functions f, g satisfying the following two conditions.

(i) R_1, f and g provide an Analogy for R.

(ii) $\mathcal{D}(f) = Sup(R)$.

Theorem 1. If R_1, f and g provide a Complete Analogy for R, then $\mathcal{D}(g) = Supterms\ (R)$.

To prove Theorem 1 note that if R_1, f and g provide a Complete Analogy for R then the Set A in condition (i) of Definition 9 becomes Sup(R). Thus, t is in $\mathcal{D}(g)$ if and only if t is in PT$\cup$ST and there is an s in Sup(R) such that Fts. From Definition 8, it immediately follows that $\mathcal{D}(g)$ = Supterms (R).

Figure 3 provides a diagrammatic representation of the notion of a Complete Analogy. Figure 3 differs from Figure 2 in that spaces A and B in Figure 2 have been expanded to correspond completely with Sup(R) and Supterms (R), respectively. However, as appropriate the ranges of these functions remain subsets of $\mathrm{Sup}(R_1) \cup \mathrm{Res}(R_1)$ and $PT_1 \cup ST_1$, respectively.

A Partial Analogy for R is an Analogy in which only some of the Suppositions are mapped into the Suppositions and Results for R_1. That is, not all of the assumptions made concerning the subject matter represented by R find corresponding Statements within R_1.

Definition 11. A Partial Analogy for Representation R is a Representation R_1, and a pair of functions f, g satisfying the following two conditions.

(i) R_1, f and g provide an Analogy for R.

(ii) $\mathcal{D}(f) \subset \mathrm{Sup}(R)$.

Theorem 2. If R_1, f and g provide a Partial Analogy for R, then

$\mathcal{D}(g) \subseteq$ Supterms (R).

The proof of Theorem 2 is straightforward and is omitted. A diagram for the notion of Partial Analogy would be exactly like Figure 2.

Our final refinement on the notion of Analogy is that of an Exact Analogy. Representation R_1, f and g provide an Exact Analogy for R just in case all Suppositions of R are mapped into Suppositions for R_1, and only Suppositions for R_1. In this sense, Exact Analogies are based on Suppositions that are exactly alike in that they only differ structurally in Terms they use.

Definition 12. An Exact Analogy for Representation R is a Representation R_1 and a pair of functions f and g satisfying the following three conditions.

(i) R_1, f and g provide an Analogy for R.

(ii) $\mathcal{D}(f) = \text{Sup}(R)$.

(iii) $\mathcal{R}(f) = \text{Sup}(R_1)$.

Theorem 3. If R_1, f and g provide an Exact Analogy for R, then R_1, f and g provide a Complete Analogy for R.

Theorem 3 follows immediately from Definitions 10 and 12.

Theorem 4. If R_1, f and g provide an Exact Analogy for R then

(i) $\mathcal{D}(g) = \text{Supterms }(R)$.

(ii) $\mathcal{R}(g) = \text{Supterms }(R_1)$.

Part (i) of Theorem 4 holds by virtue of Theorems 1 and 4. For part (ii) of Theorem 4 assume first that $t_1 \varepsilon \mathcal{R}(g)$. From part (i) this means we have a t in Supterms (R) such that $g(t)=t_1$. Since t is in Supterms (R) we know there is an $s \varepsilon \text{Sup}(R)$ such that Fts. Definition 12 now implies the existence of an s_1 in $\text{Sup}(R_1)$ such that $f(s)=s_1$. From condition (iv) of Definition 9, we have $Fg(t)f(s)$ or Ft_1s_1. But s_1 is in $\text{Sup}(R_1)$ and so $t_1 \in$ Supterms (R_1). Whence, $\mathcal{R}(g) \subseteq$ Supterms (R_1).

Now assume $t_1 \varepsilon$ Supterms (R_1). This means we have an $s_1 \varepsilon$ Sup(R_1) such that Ft_1s_1. Since f and g are both one-to-one functions, we can refer to their inverses and, in particular, to $f^{-1}(s_1)$ and $g^{-1}(t_1)$. From condition (iv), Definition 9, we know that $Fg^{-1}(t_1)f^{-1}(s_1)$ holds. Now $f^{-1}(s_1)$ is in Sup(R) by Definition 12. Whence, $g^{-1}(t_1)$ is in Supterms (R). From part (i) of Theorem 4, we know that g is defined on the set of all Supterms (R). In other words, $g(g^{-1}(t_2)) = t_1$ must be in the range of g. That is, $\mathcal{R}(g) \subseteq$ Supterms (R_1), completing the proof of Theorem 4.

The essence of Definition 12 and Theorems 3 and 4 are summarized in Figure 4.

These four types of Analogy posses some basic properties that are of interest. First, consider the notion of an Exact Analogy. Suppose R_1, f and g provide an Exact Analogy for R. Since f and g are both one-to-one, their inverses, f^{-1} and g^{-1} respectively, are also one-to-one. Thus, we can show that R, f^{-1} and g^{-1} provide an Exact Analogy for R_1. The converse result holds as well.

Theorem 5. Representation R_1 and functions f and g provide an Exact Analogy for Representation R if and only if R, f-1 and g^{-1} provide an Exact Analogy for R_1.

A result similar to Theorem 5 is not forthcoming for Analogy, Complete Analogy, or Partial Analogy. While f^{-1} and g^{-1} have the appropriate structure (one-to-one) in these cases, there is no guarantee that $\mathcal{D}(f^{-1}) \subseteq \mathrm{Sup}(R_1)$. Indeed, it is entirely possible for $\mathcal{D}(f^{-1}) = \mathcal{R}(f) \subseteq \mathrm{Res}(R_1)$. We are assuming here that R_1, f and g provide

117

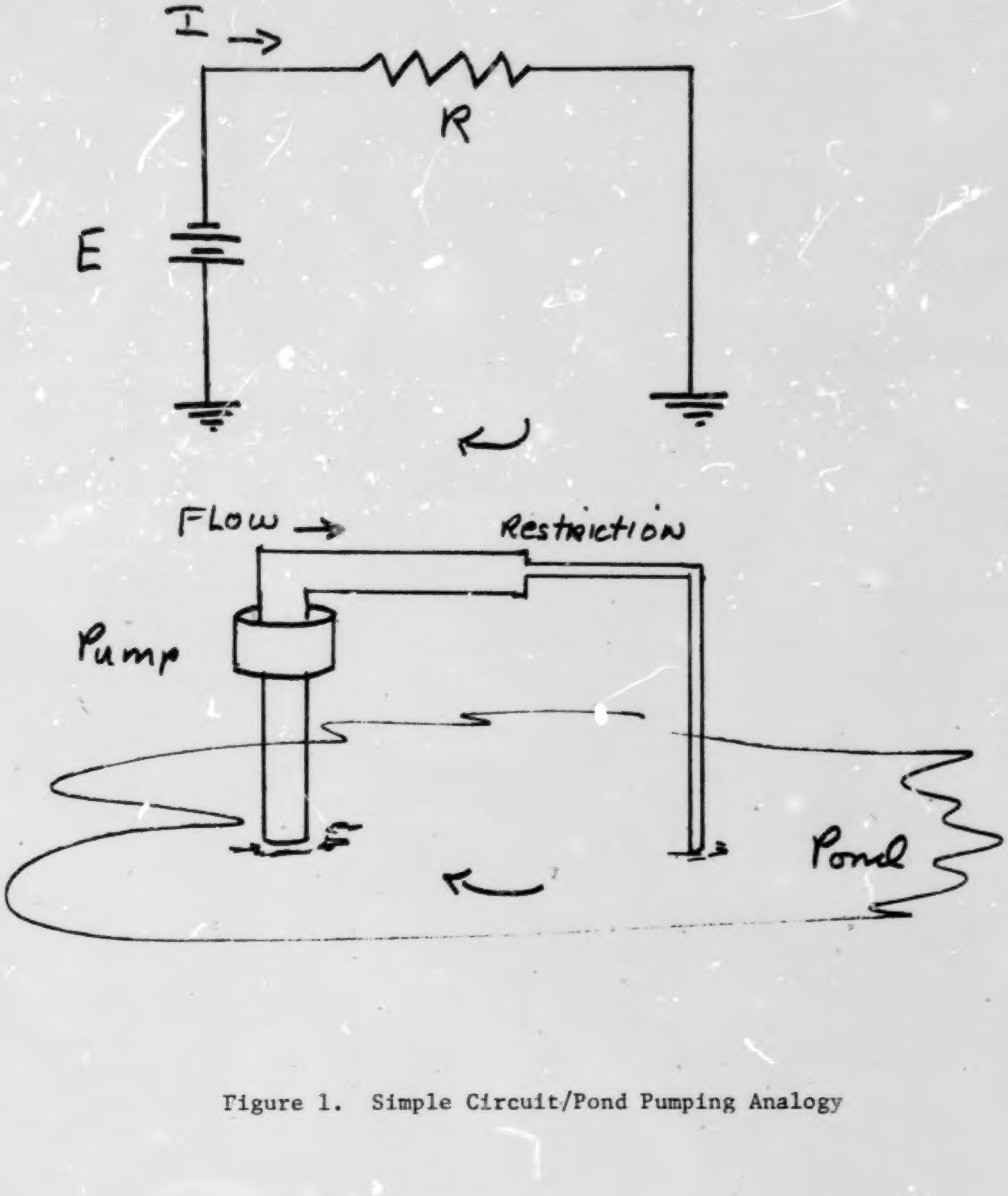

I →
R
E
FLOW →
Restriction
Pump
Pond

Figure 1. Simple Circuit/Pond Pumping Analogy

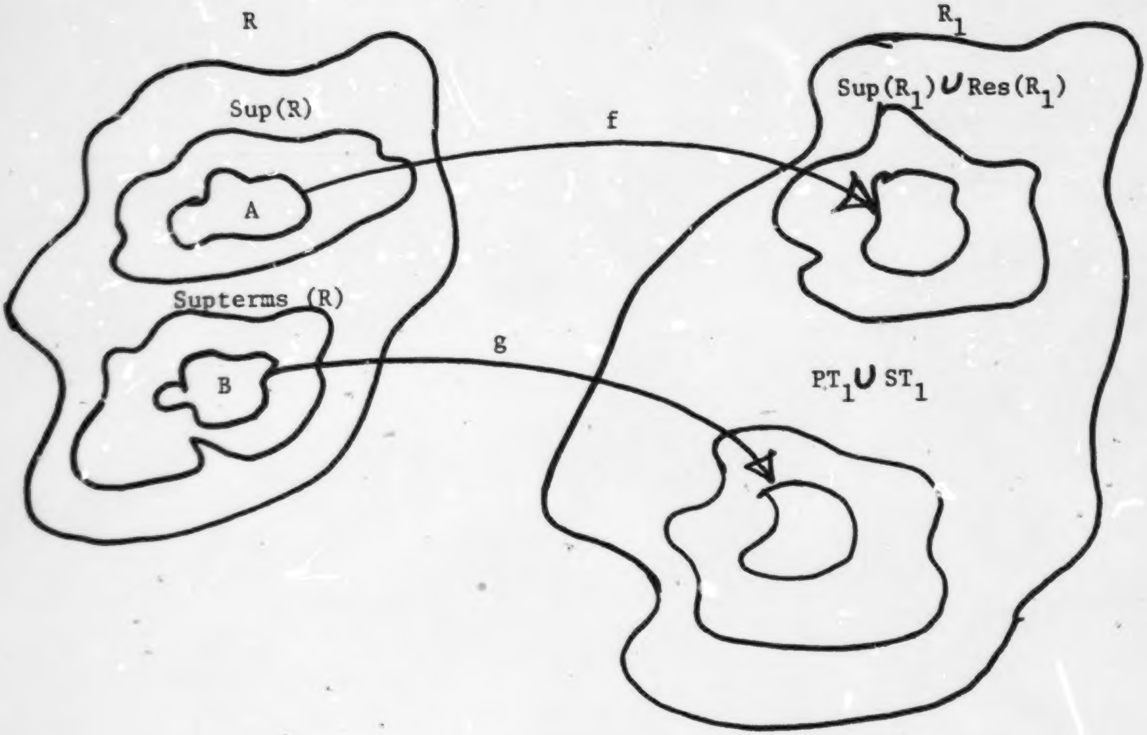

R
Sup(R)
A
Supterms (R)
B
f
g
R_1
$Sup(R_1) \cup Res(R_1)$
$PT_1 \cup ST_1$

Figure 2. R_1, f and g provide an Analogy for R

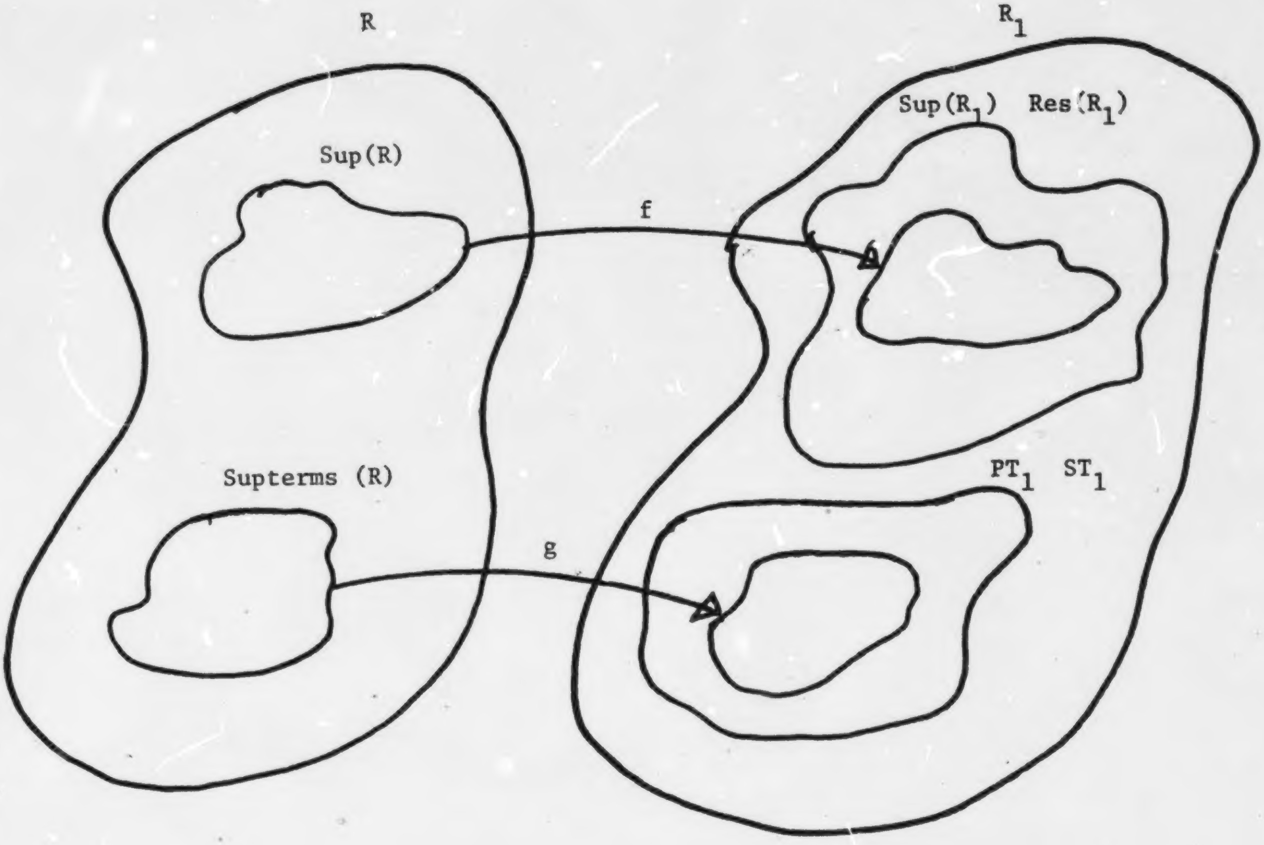

R
Sup(R)
Supterms (R)
f
g
R_1
$Sup(R_1)$
$Res(R_1)$
PT_1
ST_1

Figure 3. R_1, f and g provide a Complete Analogy for R

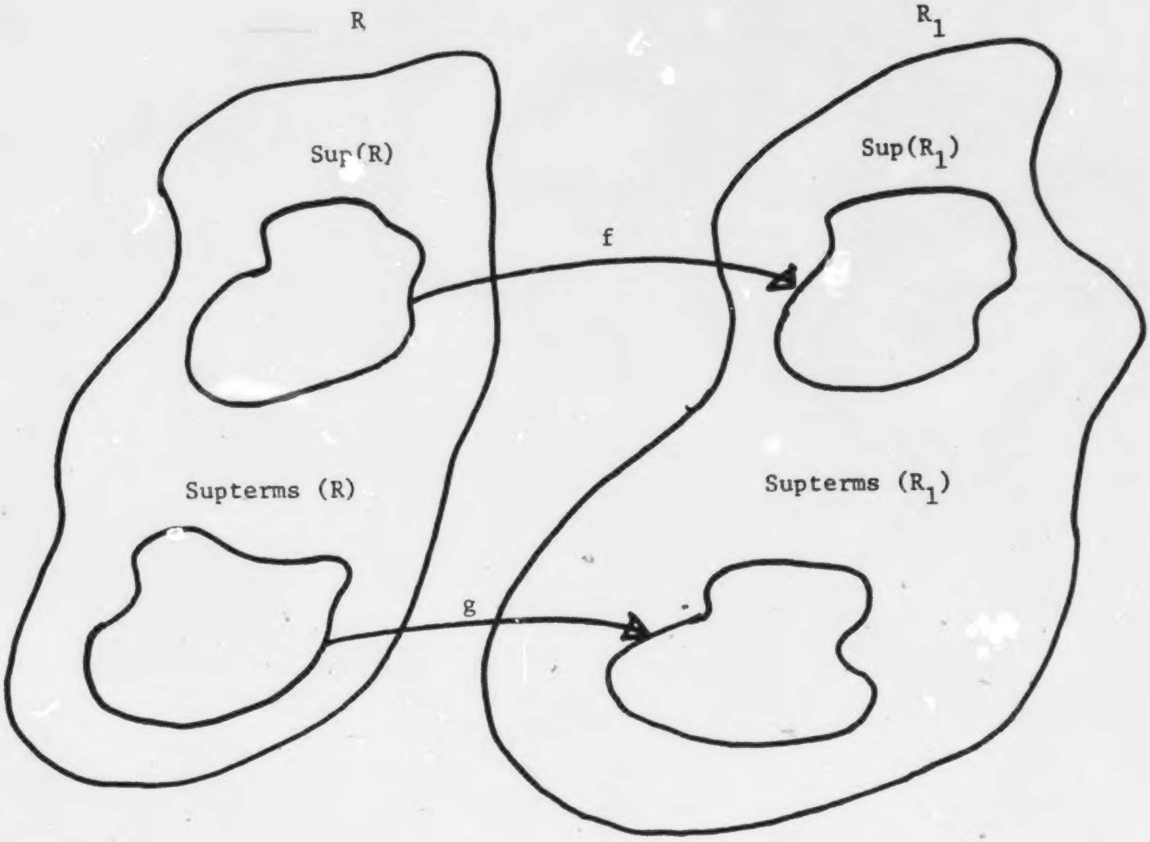

R
R1
Sup(R)
Sup(R1)
f
Supterms (R)
Supterms (R1)
g

Figure 4. R_1, f and g provide an Exact Analogy for R either an Analogy, a Complete Analogy or a Partial Analogy for R.

Now consider the case where R_1, f and g provide an Exact Analogy for R; and R_2, f_1, g_1 provide an Exact Analogy for R_1. Then R_2 along with the composite functions $f \circ f_1$ and $g \circ g_1$ provide an Exact Analogy for R. That is, the composite functions $f \circ f_1$ and $g \circ g_1$ provide the requisite mappings from R to R_2. Recall that for all s in Sup(R),

$$f \circ f_1(s) = f_1(f(s)) ,$$

and for all t in Supterms (R)

$$g \circ g_1(t) = g_1(g(t)).$$

Composites of one-to-one functions are known to be one-to-one and so $f \circ f_1$ and $g \circ g_1$ satisfy conditions (i), (ii) and (iii) of Definition 9. For condition (iv) we have both

$$Fts \leftrightarrow Fg(t)f(s)$$

and

$$Fg(t)f(s) \leftrightarrow Fg_1(g(t))f_1(f(s)).$$

Thus,

$$Fts \leftrightarrow Fg \circ g_1(t) f \circ f_1(s)$$

as required. Finally

$$\mathcal{D}(f \circ f_1) = \mathcal{D}(f) = \mathrm{Sup}(R)$$

and

$$\mathcal{R}(f \circ f_1) = \mathcal{R}(f_1) = \mathrm{Sup}(R_2)$$

and, thus, Definition 12 is satisfied as required.

We have proved the following theorem.

Theorem 6. If R_1, f and g provide an Exact Analogy for R and if R_2, f_1 and g_1 provide an Exact Analogy for R, then R_2, $f \circ f_1$ and $g \circ g_2$ provide an Exact Analogy for R.

As with theorem 5, a result similar to Theorem 6 is *not* forthcoming for Analogy, Complete Analogy or Partial Analogy. As before, the reason is that we do *not* know in these cases that $\mathcal{R}(f) \subseteq \mathrm{Sup}(R_1)$. Thus, we are not assured that the composite function $f \circ f_1$ is defined. That is, for $f \circ f_1$ to be defined, it *must* be the case that

$$\mathcal{R}(f) \subseteq \mathcal{D}(f_1).$$

But, we only know that

$$\mathcal{D}(f_1) \subseteq \mathrm{Sup}(R_1)$$

and we can*not* guarantee that

$$\mathcal{R}(f) \subseteq \mathrm{Sup}(R_1)$$

holds.

We *can* establish the following theorems.

Theorem 7. If R_1, f and g provide an Exact Analogy for R, and R_2, f_1 and g_1 provide a Complete Analogy for R_1, then R_2, $f \circ f_1$ and $g \circ g_1$ provide a Complete Analogy for R.

Theorem 8. If R_1, f and g provide an Exact Analogy for R, and R_2, f_1 and g_1 provide an Analogy (or a Partial Analogy) for R_1, then R_2, $f \circ f_1$ and $g \circ g_1$ provide an Analogy (or Partial Analogy) for R.

The difference between Theorems 7 and 8 is quite simple. In Theorem 7 every element in Sup(R) is mapped into $\mathrm{Sup}(R_2) \cup \mathrm{Res}(R_2)$ by $f \circ f_1$. In Theorem 8 only a subset of Sup(R) is mapped into this set. That is, in Theorem 8, at best,

$$\mathcal{D}(f_1) \subseteq \mathrm{Sup}(R_1).$$

Thus, only those elements, s, in Sup(R) such that

$$f(s) \in \mathcal{D}(f_1)$$

have images in $\mathrm{Sup}(R_2) \cup \mathrm{Res}(R_2)$. Similar remarks apply to Supterms (R) as well.

We will now give more concrete examples of the four types of Analogy defined so far. These examples will be based on an expansion of the electronics/pond pumping analogy discussed earlier. We will also use these developments as examples for the definitions to be introduced below that serve to compare relative strengths of analogous Statements.

We let R be an Axiomatically Structured Representation for a part of elementary electronics. Since we need only be concerned here with the Suppositions for R, we will not be concerned with developing a complete characterization of R. That is, we will only specify those portions of R that are relevant to our purposes. Similar remarks apply to R_1, the Representation for the pond pumping subject matter. These examples will be relied upon and referred to throughout the remainder of this chapter.

In these examples we have chosen to represent resistance by "R", "R_1", "R_2", etc., in order to conform to standard notational conventions. The context in which "R", etc., occur will always clearly distinguish whether a Representation, also referred to by "R" etc., or resistance is being referenced. Whence, no confusion should arise from our notational conventions.

We take three Statements as being the complete set of Suppositions for R.

s_1 : $E = I \times R$.

s_2 : In a series circuit $R_T = R_1 + R_2$.

s_3 : In a parallel circuit R_T is proportional to $R_1 \times R_2$.

The meanings of s_1, s_2 and s_3 should be clear where R_T is total resistance and R_1, R_2 denote individual resistance values. While we recognize that s_1, s_2 and s_3 are valid assertions, as stated they appear slightly out of context. However, this highly superficial account of basic electronic principles will serve our needs. Which Terms in s_1, s_2 and s_3 are Primary and which are Secondary need not concern us. What is important for us is that

$$\text{Supterms }(R) = \{\text{series circuit}, \text{parallel circuit}, E, I, R, R_T, R_1, R_2\}$$

Notice that " = ", " × " and "proportional to" do not appear in Supterms (R). This is tantamount to including these terms in L, the underlying language for R. That is, the meaning of these terms is assumed for R.

As Suppositions in R_1 we take the following three Statements.

x_1 : $Pr = F\ell \times Re$.

x_2 : In a series pipe Re_T is proportional to $Re_1 + Re_2$.

125

x_3 : In a parallel pipe

$$Re_T = \frac{Re_1 \times Re_2}{Re_1 + Re_2} .$$

Representation R_1 is assumed to be based on L, the underlying language for R. Whence, we have

$$\text{Supterms } (R_1) = \{\text{series pipe, parallel pipe, } Pr, F\ell, Re, Re_T, Re_1, Re_2\}.$$

We add to R_1 one other Statement

x_4 : In a series pipe

$$Re_T = Re_1 + Re_2 ,$$

and we assume that x_4 is a Result in R_1. That is, $x_4 \in Res(R_1)$.

As our first example, we construct two functions, f and g, that with R_1 result in an Analogy for R. We define f as follows.

$$f(s_1) = x_1.$$

$$f(s_2) = x_4.$$

The function g is defined next.

$$g(E) = Pr.$$

$$g(I) = F\ell.$$

$$g(R) = Re.$$

$$g\ (\text{series circuit}) = \text{series pipe}.$$

$$g(RT) = Re_T.$$

$$g(R_1) = Re_1.$$

$$g(R_2) = Re_2.$$

It should be clear that g satisfies condition (iii) of Definition 9. The functions f and g also satisfy condition (iv) of Definition 9. For example, consider s_1.

We have

$$FEs_1$$

$$FIs_1$$

and

$$FRs_1.$$

A consideration of x_4 shows that we have

$$FPrx_4$$

$$FF\ell x_4$$

and

$$FRex_4.$$

From the definition of f and g this means that we have

$$Fg(E)f(s_1)$$

$$Fg(I)f(s_1)$$

and

$$Fg(R)f(s_1)$$

showing the validity of condition (iv) for s_1. A similar argument establishes the validity of condition (iv) for s_2; and hence, condition (iv) has been shown to hold.

For A in condition (ii) of Definition 9 we have

$$A = \{s_1, s_2\}.$$

The range of f, $\mathcal{R}(f)$, is the set

$$\{x_1, x_4\}$$

which is a subset of

$$\mathrm{Sup}(R_1) \cup \mathrm{Res}(R_1).$$

Thus, since f is one-to-one, Condition (i) is satisfied. In condition (ii) of Definition 9

$$B = \{E, I, R, \text{series circuit}, R_T, R_1, R_2\}.$$

An inspection of the definition of g shows that $\mathcal{D}(g)=B$ as required. Since $\mathcal{R}(g) \subseteq PT_1 \cup ST_1$ we see that condition (ii) is satisfied.

Thus, we have shown that R_1 with f and g provide an Analogy for R. More precisely, R_1, f and g provide a Partial Analogy for R. This holds by virtue of the fact that $s_3 \in \mathrm{Sup}(R)$ but $s_3 \notin \mathcal{D}(f)$. That is, $\mathcal{D}(f)$ is a proper subset of Sup(R) as required in Definition 11.

By modifying f and g, but keeping R constant, we can construct a Complete Analogy for R. We must first extend f so that its domain includes s_3:

$$f(s_3) = x_3.$$

We must also extend g to include in its domain Terms appearing in s_3 not already in the domain of g:

$$g\text{ (parallel circuit)} = \text{parallel pipe.}$$

With these extensions it should be clear that R_1, f and g provide an Analogy for R. In addition, since $\mathcal{D}(f) = \text{Sup}(R)$, condition (ii) of Definition 11 is satisfied. Whence, R_1, f and g now provide a Complete Analogy for R. Notice also that consistent with Theorem 1, $\mathcal{D}(g) = \text{Supterms}(R)$, as expected.

We can also construct an Exact Analogy for R. To do so, we take g as defined above--in extended form in the Complete Analogy example. The function f must be redefined:

$$f(s_1) = x_1$$
$$f(s_2) = x_2$$
$$f(s_3) = x_3.$$

Our old function f associated s_2 with x_4. Since x_4 is a Result for R_1, the old version of f had a range that did not correspond with $\text{Sup}(R_1)$. The new definition of f above eliminates this problem by associating s_2 with x_2, a member of $\text{Sup}(R_1)$. That is, we have redefined f so that

$$\mathcal{R}(f) = \text{Sup}(R_1).$$

Since

$$\mathcal{D}(f) = \text{Sup}(R),$$

we see that R_1, f and g now provide an Exact Analogy for R as defined in Definition 12.

This completes our discussion of the purely structural characteristics of Analogies. We now turn to the problem of comparing the derivational strengths of Analogies. Let R_1, f and g provide an Analogy for R. Suppose s is both in Sup(R) and in $\mathcal{D}(f)$. This means that s has an image under f, f(s), in $\mathrm{Sup}(R_1) \cup \mathrm{Res}(R_1)$. We will refer to f(s) as the analogous interpretation (in R_1) of s. We are interested in comparing all s in $\mathcal{D}(f)$ to their analogous interpretations in R_1. We will distinguish three types of Analogies in this way: Proper, Approximate and Enveloping.

Let us begin with Proper Analogies. The idea behond a Proper Analogy can be stated in terms of the electronics and pumping example. Representations for these subject matters are given above. Consider two functions, f and g, where f and g are completely defined as follows:

$$f(s_1) = x_1 \ ,$$

$$g(E) = Pr \ ,$$

$$g(I) = F\ell \ ,$$

and

$$g(R) = Re \ .$$

R_1, f and g provide an Analogy, in fact, a Partial Analogy, for R. Now notice that $f(s_1)$, x_1, arises from a direct substitution of images, under g, of Terms in s_1 for those Terms. That is, we can transform s_1 into x_1 as follows:

$$\begin{array}{ll} s_1 : & E = I \times R \\ & g(E) = g(I) \times g(R) \\ x_1 : & Pr = F\ell \times Re. \end{array}$$

A Proper Analogy will be one that exhibits this characteristic for all s in the domain of f. That is, an Analogy is Proper if for all Statements with analogous interpretations, the analogous interpretations arise from a direct substitution of images (under g) of Terms.

To make these notions more precise, we introduce a general method of referring to Statements arising from substitutions of this type.

Definition 13. Let R_1, f and g provide an Analogy for R. For all $s\varepsilon\mathcal{D}(f)$

$$\mathrm{Subs}(s/g(t),t)$$

denotes the result of substituting in Statement s, g(t) for *every* occurrence of *every* term in $\mathcal{D}(g)$.

It is important to notice that in forming Subs(s/g(t),t) *every* Term, t, such that Fts is replaced by g(t). Also *every* occurrence of such a Term is also replaced. We can show that the result of substitution of this type is a Statement within PS_1.

Theorem 9. Let R_1, f and g provide an Analogy for R. If $s\varepsilon\mathcal{D}(f)$, then

$$\mathrm{Subs}(s/g(t),t)\ \varepsilon\ PS_1.$$

To prove Theorem 9, let R_1, f and g provide an Analogy for R and let $s\varepsilon\mathcal{D}(f)$. This means that $s\varepsilon\mathrm{Sup}(R)$. The only terms in s are either from L or from $PT_1 \cup ST_1$. In the latter case, such a Term, t, is in Supterms (R) which implies $t\varepsilon\mathcal{D}(g)$. Therefore, g(t) is defined for every Term in s not in the lexicon of L. Let

$$s_1 = \mathrm{Subs}(s/g(t),t)$$

and notice that the only terms in s_1 are either in the lexicon of L or in $PT_1 \cup ST_1$. But $L=L_1$ since R_1, f, g are an Analogy for R, and so every term in s_1 is either in L_1 or in $PT_1 \cup ST_1$ But since g(t) and t have the same logical structure, which means they function syntactically in the same way, it follows that s_1 is a well formed expression in L_1^1, L_1 extended. Whence, $s_1 \varepsilon PS_1$ as required.

Notice also that g is one-to-one and has a one-to-one inverse, g^{-1}. This means that if we start with the Statement arising from

$$\text{Subs}(s/g(t),t)$$

and replace all occurrences in this Statement of g(t) with $g^{-1}(g(t))$, then we end up with s. In other words

$$\text{Subs}(\text{Subs}(s/g(t),t)$$
$$/g^{-1}(g(t)),g(t)) = s.$$

We can now state the precise definition of a Proper Analogy.

Definition 14. A Proper Analogy for a Representation R is a Representation R_1 and a pair of functions f, g satisfying the following two conditions:

(i) R_1, f and g provide an Analogy for R.

(ii) For all s in $\mathcal{D}(f)$

$$\text{Sub}(s/g(t),t) = f(s).$$

As another example of a Proper Analogy consider R and R_1 as in the electronics/pond example, and define f and g as follows:

$f(s_1) = x_1$.

$f(s_2) = x_4$.

$g(E) = Pr$.

$g(I) = F\ell$.

$g(R) = Re$.

g (series circuit) = series pipe

$g(R_T) = Re_T$

$g(R_1) = Re_1$

$g(R_2) = Re_2$

Notice that $Subs(s_1/g(t),t)$ becomes

$$Subs(E = I \times R/g(t),t),$$

or

$$g(E) = g(I) \times g(R)$$

which is x_1, or $f(s_1)$:

$$Pr = F\ell \times Re.$$

Also $Subs(s_2/g(t),t)$ becomes

Subs (in a series circuit $R_T = R_1 + R_2$ $/g(t),t)$

or

In a g (series circuit)

$g(R_T) = g(R_1) + g(R_2)$.

From the definition of g we have

In a series pipe

$Re_T = Re_1 + Re_2$

which is x_4 or $f(s_2)$ as required.

Complete Analogies, Partial Analogies and Exact Analogies can also be Proper as asserted in the next definition.

Definition 15. A Proper Complete, Partial or Exact Analogy for a Representation R is a Representation R_1 and a pair of functions f, g satisfying the folowing two conditions.

(i) R_1, f and g provide, respectively, a Complete, Partial or Exact Analogy for R.

(ii) For all s in $\mathcal{D}(f)$

$$\text{Subs}(s/g(t),t) = f(s).$$

The last example given above is an example of a Proper Partial Analogy since $\mathcal{D}(f)$ is a proper subset of Sup(R). That is, $s_3\mathcal{D}(f)$. In the case of the electronics/pond example, it is not possible to construct either a Proper Complete Analogy or a Proper Exact Analogy. In either of these cases, s_3 does not have a suitable image among x_1, x_2, x_3 and x_4. If we extend g as last defined to include

g (parallel circuit) = parallel pipe,

then $\text{Subs}(s_3/g(t),t)$ becomes

In a parallel circuit

Re_T is proportional to $Re_1 \times Re_2$.

134

This Statement is not found in

$$S_1 = \{x_1, x_2, x_3, x_4\}.$$

Suppose we label this Statement as x_5 and include it in $\mathrm{Res}(R_1)$. We now will have a Proper Complete Analogy if we extend our last defined function f to include:

$$f(s_3) = x_5.$$

Now $\mathcal{D}(f) = \mathrm{Sup}(R)$ as required.

Notice, however, that we still do not have a Proper Exact Analogy. The reason for this is that

$$\mathcal{R}(f) \neq \mathrm{Sup}(R_1)$$

since

$$f(s_3) = x_5 \notin \mathrm{Sup}(R_1).$$

To have R_1, f and g provide a Proper Exact Analogy requires us to place x_5 in $\mathrm{Sup}(R_1)$.

The next theorem follows immediately from Theorem 3 and Definition 15.

Theorem 10. If R_1, f and g provide a Proper Exact Analogy for R, then R_1, f and g provide a Proper Complete Analogy for R.

The next theorem can be seen to hold by virtue of Theorem 5 and Definition 15.

Theorem 11. Representation R_1 and functions f and g provide a Proper Exact Analogy for Representation R if and only if R, f^{-1} and g^{-1} provide a Proper Exact Analogy for R_1.

A transitivity property similar to that expressed in Theorem 6 also applies to Proper Exact Analogies.

Theorem 12. If R_1, f and g provide a Proper Exact Analogy for R, and if R_2, f_1 and g_1 provide a Proper Exact Analogy for R_1, then R_2, $f \circ f_1$ and $g \circ g_1$ provide a Proper Exact Analogy for R.

The next two theorems are entirely similar to Theorems 7 and 8, respectively.

Theorem 13. If R_1, f and g provide a Proper Exact Analogy for R, and R_2, f_1 and g_1 provide a Proper Complete Analogy for R_1, then R_2, $f \circ f_1$ and $g \circ g_1$ provide a Proper Complete Analogy for R.

Theorem 14. If R_1, f and g provide a Proper Exact Analogy for R, and R_2, f_1 and g_1 provide a Proper Analogy (or Proper Partial Analogy) for R_1, then R_2, $f \circ f_1$ and $g \circ g_1$ provide a Proper Analogy (or Proper Partial Analogy) for R.

We have now defined Proper Analogies. What do we mean by Approximate Analogies? An Analogy is Approximate if the analogous Statements for s in $\mathcal{D}(f)$ are (perhaps) not as strong as, but are implied by, Subs(s/g(t),t). That is, the Statement in R_1, corresponding to Supposition, s, for R, is not as strong a Statement abour R_1's subject matter as s is about R's

136

subject matter. As an example, we return to the electronics/pond Representations, R and R_1. The functions f and g are completely characterized as follows.

$$f(s_2) = x_2$$

$$g\ (\text{series circuit}) = \text{series pipe}$$

$$g(R_T) = Re_T$$

$$g(R_1) = Re_1$$

$$g(R_2) = Re_2$$

In this case, R_1, f and g provide an Analogy (Partial) for R. Notice that

$$\text{Subs}(s_2/g(t),t)$$

becomes

In a series pipe

$$Re_T = Re_1 + Re_2$$

which is unequal to x_2. The Statement x_2 only asserts that

In a series pipe

Re_T is proportional to $Re_1 + Re_2$.

Thus, R_1, f and g do not provide a Proper Analogy or a Proper Partial Analogy for R.

But notice that

$$\text{Subs}(s_2/g(t),t)$$

implies x_2 or $f(s_2)$. That is, we can prove that

$$\mathrm{Subs}(s_2/g(t),t) \rightarrow f(s_2)$$

holds. Such a proof will rely only on elements in R_1 so that the proof can be carried out in PS_1. That is, the defining criteria for an Approximate Analogy is:

$$PS_1 \models \mathrm{Subs}(s_2/g(t),t) \rightarrow f(s_2).$$

Our formal definition of Approximate Analogy is as follows.

Definition 16. An Approximate Analogy for a Representation R is a Representation R_1 and a pair of functions f, g satisfying the following two conditions.

(i) R_1, f and g provide an Analogy for R.

(ii) For all s in $\mathcal{D}(f)$

$$PS_1 \models \mathrm{Subs}(s/g(t),t) \rightarrow f(s).$$

Definition 17. An Approximate Complete, Partial or Exact Analogy for a Representation R is a Representation R_1 and a pair of functions f and g satisfying the following two conditions.

(i) R_1, f and g provide, respectively, a Complete, Partial or Exact Analogy for R.

(ii) For all s in $\mathcal{D}(f)$

$$PS_1 \models \mathrm{Subs}(s/g(t),t) \rightarrow f(s).$$

We have already given an example of an Approximate Partial Analogy. Using R_1 with $x_5 \in \mathrm{Res}(R_1)$ as defined above, we can give an example of an Approximate Complete Analogy for the Electronics Representation.

$f(s_1) = x_1$

$f(s_2) = x_2$

$f(s_3) = x_5$

$g(E) = Pr$

$g(I) = F\ell$

$g(R) = Re$

g (series circuit) = series pipe

$g(R_T) = Re_T$

$g(R_1) = Re_1$

$g(R_2) = Re_2$

g (parallel circuit) = parallel pipe

Clearly, R_1, f and g constitute a Complete Analogy. To see that the Complete Analogy is also Approximate, notice that

$$\mathrm{Subs}(s_1/g(t),t) = x_1\ ,$$

$$\mathrm{Subs}(s_2/g(t),t) = \text{In a series pipe}\quad Re_T = Re_1 + Re_2\ ,$$

and

$$\mathrm{Subs}(s_3/g(t),t) = x_5.$$

As facts from Logic we know

$$PS_1 \models x_1 \rightarrow x_1$$

and

$$PS_1 \models x_5 \rightarrow x_5.$$

Thus,

$$PS_1 \models \mathrm{Subs}(s_1/g(t),t) \rightarrow f(s_1)$$

and

$$PS_1 \models \mathrm{Subs}(s_3/g(t),t) \rightarrow f(s_3).$$

Notice also that we can prove that

In a series pipe

$$Re_T = Re_1 + Re_2$$

implies that

In a series pipe Re_T is proportional
to $Re_1 + Re_2$.

That is, we have

$$PS_1 \models \mathrm{Subs}(s_2/g(t),t) \rightarrow f(s_2)$$

which establishes the fact that R_1, f and g provide an Approximate Complete Analogy for R.

If we are willing to count x_5 as a member of $\mathrm{Sup}(R_1)$, then the Analogy becomes an Approximate Exact Analogy. In the above form where $x_5 \in \mathrm{Res}(R_1)$, however, the Analogy is not an Exact Analogy.

The above example points out an interesting relationship between Approximate and Proper Analogies. Suppose R_1, f and g constitute a Proper Analogy for R. That is, for all s in $\mathcal{D}(f)$

$$\mathrm{Subs}(s/g(t),t) = F(s).$$

For any Statement s we know

$$PS_1 \models s \rightarrow s,$$

or

$$PS_1 \models f(s) \rightarrow f(s).$$

Thus, for any s in $\mathcal{D}(f)$ we have

$$PS_1 \models \mathrm{Subs}(s/g(t),t) \rightarrow f(s).$$

This establishes the following theorem.

Theorem 15. If R_1, f and g provide a Proper Analogy, Proper Complete Analogy, Proper Partial Analogy or a Proper Exact Analogy, for R, then R_1, f and g provide, respectively, an Approximate Analogy, Approximate Complete Analogy, Approximate Partial Analogy or an Approximate Exact Analogy for R.

We state without proof four other interesting theorems.

Theorem 16. If R_1, f and g provide an Approximate Exact Analogy for R, then R_1, f and g provide an Approximate Complete Analogy for R.

Theorem 17. If R_1, f and g provide an Approximate Exact Analogy for R, and if R_2, f_1 and g_1 provide an Approximate Exact Analogy for R_1, then R_2, $f \circ f_1$ and $g \circ g_1$ provide an Approximate Exact Analogy for R.

Theorem 18. If R_1, f and g provide an Approximate Exact Analogy for R, and if R_2, f_1 and g_1 provide an Approximate Complete Analogy for R_1, then R_2, $f \circ f_1$ and $g \circ g_1$ provide an Approximate Complete Analogy for R.

Theorem 19. If R_1, f and g provide an Approximate Exact Analogy for R, and if R_2, f_1 and g_1 provide an Approximate Analogy (or Approximate Partial Analogy) for R_1, then R_2, $f \circ f_1$ and $g \circ g_1$ provide an Approximate Analogy (or Approximate Partial Analogy) for R.

We can also establish six relationships between various kinds of Proper and Approximate Analogies. The following three theorems summarize these relationships. In the antecedent of each theorem, one must either read the unbracketed word preceding each bracket or one must substitute each bracketed word for the preceding word. These options may not be mixed. For example, Theorem 20 (Theorems 21 and 22, as well) really asserts two theorems. The first is a relation between

Proper Exact Analogies

and, next,

Approximate Exact Analogies

(the unbracketed version). The second (bracketed version) asserts a relation between

Approximate Exact Analogies

and, next,

Proper Exact Analogies.

Similar remarks apply to Theorems 21 and 22.

Theorem 20. If R_1, f and g provide a Proper [Approximate] Exact Analogy for R, and if R_2, f_1 and g_1 provide an Approximate [Proper] Exact Analogy for R_1, then R_2, $f \circ f_1$ and $g \circ g_1$ provide an Approximate Exact Analogy for R.

Theorem 21. If R_1, f and g provide a Proper [Approximate] Exact Analogy for R, and if R_2, f_1 and g_1 provide an Approximate [Proper] Complete Analogy for R_1, then R_2, $f \circ f_1$ and $g \circ g_1$ provide an Approximate Complete Analogy for R.

Theorem 22. If R_1, f and g provide a Proper [Approximate] Exact Analogy for R, and if R_2, f_1 and g_1 provide an Approximate [Proper] Analogy (or Partial Analogy) for R, then R_2, $f \circ f_1$ and $g \circ g_1$ provide an Approximate Analogy (or Partial Analogy) for R.

The final type of Analogy to be discussed is that of an Enveloping Analogy. The idea behind Enveloping Analogy is that Analogous statements for Suppositions in R are stronger than or envelop their corresponding Suppositions. In the electronics/pond example let f be defined as

$$f(s_3) = x_3 ;$$

and let g be defined as

g (parallel circuit) = parallel pipe

$g(R_T) = Re_T$

$g(R_1) = Re_1$

$g(R_2) = Re_2$.

Consider $Subs(s_3/g(t),t)$ which becomes

In a parallel pipe

Re_T is proportional to $Re_1 \times Re_2$.

Comparing this Statement to f(s) we see that it is implied by f(s). That is, we have:

$$PS_1 \models f(s_3) \rightarrow Subs(s_3/g(t),t).$$

This means that $f(s_3)$ is stronger than $Subs(s_3/g(t),t)$. Accordingly, we take this condition as our defining criteria for Enveloping Analogies.

Definition 18. An Enveloping Analogy for a Representation R is a Representation R_1 and a pair of functions f, g satisfying the following two conditions.

(i) R_1, f and g provide an Analogy for R.

(ii) For all s in $\mathcal{D}(f)$

$$PS_1 \models f(s) \rightarrow Subs(s/g(t),t).$$

Definition 19. An Enveloping Complete, Partial or Exact Analogy for a Representation R is a Representation R_1 and a pair of functions f and g satisfying the following two conditions.

(i) R_1, f and g provide, respectively, a Complete, Partial or Exact Analogy for R.

(ii) For all s in $\mathcal{D}(f)$

$$PS_1 \models f(s) \rightarrow Subs(s/g(t),t).$$

We have already given an example of an Enveloping Analogy (or an Enveloping Partial Analogy). For an Enveloping Complete Analogy let

$$f(s_1) = x_1$$
$$f(s_2) = x_4$$
$$f(s_3) = x_3$$

$g(E) = Pr$

$g(I) = F\ell$

$g(R) = Fe$

g (series circuit) = series pipe

$g(R_T) = Re_T$

$g(R_1) = Re_1$

$g(R_2) = Re_2$

g (parallel circuit) = parallel pipe

Notice that

$$\mathrm{Subs}(s_1/g(t),t) = x_1.$$

Since

$$PS_1 \models x_1 \rightarrow x_1$$

and since $f(s_1) = x_1$, we see that s_1 satisfies condition (ii) of Definition 19.

For $\mathrm{Subs}(s_2/g(t),t)$ we have

In a series pipe

$$Re_T = Re_1 + Re_2.$$

This expression is identical to x_4 and since

$$PS_1 \models x_4 \rightarrow x_4$$

condition (ii) is seen to hold for s_2. For s_3, $\mathrm{Subs}(s_3/g(t),t)$ is

In a parallel pipe

Re_T is proportional to $Re_1 \times Re_2$.

Now this Statement is implied by

$$Re_T = \frac{Re_1 \times Re_2}{Re_1 + Re_2}$$

which is x_3 of $f(s_3)$. That is,

$$PS_A \models f(s_3) \rightarrow Subs(s_3/g(t),t).$$

To transform this example into an Enveloping Exact Analogy we must include x_4 in $Sup(R_1)$.

We now state some interesting relationships between Enveloping Analogies and other kinds of Analogies.

Theorem 23. If R_1, f and g provide a Proper Analogy, Proper Complete Analogy, Proper Partial Analogy or a Proper Exact Analogy, then R_1, f and g provide, respectively, an Enveloping Analogy, Enveloping Complete Analogy, Enveloping Partial Analogy, or an Enveloping Exact Analogy for R.

Theroem 24. If R_1, f and g provide an Enveloping Exact Analogy for R, then R_1, f and g provide an Enveloping Complete Analogy for R.

Theorem 25. If R_1, f and g provide an Enveloping Exact Analogy for R, then R, f^{-1} and g^{-1} provide an Approximate Exact Analogy for R_1.

Theorem 26. If R_1, f and g provide an Approximate Exact Analogy for R, then R, f^{-1} and g^{-1} provide an Enveloping Exact Analogy for R_1.

Combining Theorems 25 and 26 yields the following theorem.

Theorem 27. Representation R_1 and functions g, f provide an Enveloping Exact Analogy for Representation R if and only if R, f^{-1} and g^{-1} provide an Approximate Exact Analogy for R.

Theorem 28. If R_1, f and g provide an Enveloping Exact Analogy for R, and if R_2, f_1 and g_1 provide an Enveloping Exact Analogy for R_1, then R_2, $f \circ f_1$ and $g \circ g_1$ provide an Enveloping Exact Analogy for R.

Theorem 29. If R_1, f and g provide an Enveloping Exact Analogy for R, and if R_2, f_1 and g_1 provide an Enveloping Complete Analogy for R_1, then R_2, $f \circ f_1$ and $g \circ g_1$ provide an Enveloping Complete Analogy for R.

Theorem 30. If R_1, f and g provide an Enveloping Exact Analogy for R, and if R_2, f_1 and g_1 provide an Enveloping Analogy (or Enveloping Partial Analogy) for R_1, then R_2, $f \circ f_1$ and $g \circ g_1$ provide an Enveloping Analogy (or Enveloping Partial Analogy) for R.

The bracketed portions of the next theorems are intended to be interpreted as in Theorems 20, 21 and 22.

Theorem 31. If R_1, f and g provide a Proper [Enveloping] Exact Analogy for R, and if R_2, f_1 and g_1 provide an Enveloping [Proper] Exact Analogy for R_1, then R_2, $f \circ f_1$ and $g \circ g_1$ provide an Enveloping Exact Analogy for R.

Theorem 32. If R_1, f and g provide a Proper [Enveloping] Exact Analogy for R, and if R_2, f_1 and g_1 provide an Enveloping [Proper] Complete Analogy for R_1, then R_2, $f \circ f_1$ and $g \circ g_1$ provide an Enveloping Complete Analogy for R.

Theorem 33. If R_1, f and g provide a Proper [Enveloping] Exact Analogy for R, and if R_2, f_1 and g_1 provide an Enveloping [Proper] Analogy (or Partial Analogy) for R, then R_2, $f \circ f_1$ and $g \circ g_1$ provide an Enveloping Analogy (or Partial Analogy) for R.

In the above developments we have provided a great many statements (theorems) asserting elementary facts about, and relationships between, various kinds of Analogies. It will be helpful to summarize this data in tabular form. In particular, Table 1 provides a summary of asserted reflexive properties and Table 2 provides a summary of asserted transitive properties. Table 3 provides additional relational information not forthcoming from Tables 1 and 2. Table 2 can be interpreted as follows: If an Analogy of the type in column 2 is followed by an analogy of the type in column 3, then the resultant (composite) type is the type listed in column 4. For example, the last line means an Enveloping Partial Analogy (from R to R_1) followed by a Proper Partial Analogy (R_1 to R_2) results in an Enveloping Partial Analogy (R to R_2).

Notice that only Exact Analogies appear on the left (column 2) in this schema. Notice also that in combining analogies in this way, Approximate Analogies never follow Enveloping Analogies; nor do Enveloping Analogies follow Approximate Analogies. A close look at the definitions of these concepts show that this must be the case due to the implication that may hold between elements. That is, from s_1 implies s_2 (s_2 approximates s_1) and from s_3 implies s_2 (s_3 envelops s_2) we cannot infer any relationship between s_1 ans s_3. Similarly, knowing that s_2 implies s_1 (s_2 envelops s_1) and s_2 implies s_3 (s_3 approximates s_2) we can infer nothing about the relationship between s_1 and s_3.

It is important to realize that we have characterized Analogies solely in terms of Suppositions and subject-matter-related Terms appearing in Suppositions. Analogies are independent of Definitions and Results concerning subject matters. This is appropriate as we now suggest. We have shown that all Definitions are eliminable from a Representation (Chapter I, Theorem 2). We can turn this around and assert that given the Primary Terms we can generate all required Definitions. That is, given some type of an Analogy, we can proceed to define within the Analogy Concepts that are Analogous to interesting Concepts within the original subject matter. More specifically, g maps a set of Terms from R into R_1. Any Definition in R Definable only with Terms in the domain of g will have an Analogous Definition formulated solely with Terms in the range of g. Thus, having the Analogies for (some) Suppositions and their Terms can result in Analogies for other Definitions in the subject matter.

Similarly, consider Results in the original subject matter. We can show that any Result is provable on the basis of Suppositions only. Thus, to map Suppositions from R into Suppositions or Results in R_1 is sufficient to guarantee us that Analogous interpretations of Results are

Table 1. Reflexive Properties for Analogies

Theorem	Type of Analogy for R Provided by R_1, f and g	Type of Analogy for R_1 provided by R, F^{-1}, g^{-1}
5	Exact	Exact
11	Proper Exact	Proper Exact
25	Enveloping Exact	Approximate Exact
26	Approximate Exact	Enveloping Exact

[illegible] Transitive Properties for Analogies

Theorem	Type of Analogy for K provided by K_1, f and g	Type of Analogy for K_1 provided by K_2, f_1 and g_1	Type of Analogy for K provided by K_2, $f \circ f_1$ and $g \circ g_1$
6	Exact	Exact	Exact
7	Exact	Complete	Complete
8	Exact	Analogy	Analogy
9	Exact	Partial	Partial
12	Proper Exact	Proper Exact	Proper Exact
13	Proper Exact	Proper Complete	Proper Complete
14	Proper Exact	Proper	Proper
14	Proper Exact	Proper Partial	Proper Partial
17	Approximate Exact	Approximate Exact	Approximate Exact
18	Approximate Exact	Approximate Complete	Approximate Complete
19	Approximate Exact	Approximate	Approximate
19	Approximate Exact	Approximate Partial	Approximate Partial
20	Proper Exact	Approximate Exact	Approximate Exact
20	Approximate Exact	Proper Exact	Approximate Exact
21	Proper Exact	Approximate Complete	Approximate Complete
21	Approximate Exact	Proper Complete	Approximate Complete
22	Proper Exact	Approximate	Approximate
22	Proper Exact	Approximate Partial	Approximate Partial
22	Approximate Exact	Proper	Approximate
22	Approximate Exact	Proper Partial	Proper Partial
28	Enveloping Exact	Enveloping Exact	Enveloping Exact
29	Enveloping Exact	Enveloping Complete	Enveloping Complete
30	Enveloping Exact	Enveloping	Enveloping
30	Enveloping Exact	Enveloping Partial	Enveloping Partial
31	Proper Exact	Enveloping Exact	Enveloping Exact
31	Enveloping Exact	Proper Exact	Enveloping Exact
32	Proper Exact	Enveloping Complete	Enveloping Complete
32	Enveloping Exact	Proper Complete	Enveloping Complete
33	Proper Exact	Enveloping	Enveloping
33	Proper Exact	Enveloping Partial	Enveloping Partial
33	Enveloping Exact	Proper	Enveloping
33	Enveloping Exact	Proper Partial	Enveloping Partial

Table 3. Implied Relationships between Types of Analogies.

Theorem	Type of Analogy for R provided by R_1, f and g	Implied type of Analogy provided for R by R_1, f and g
3	Exact	Complete
10	Proper Exact	Proper Complete
15	Proper	Approximate
15	Proper Complete	Approximate Complete
15	Proper Partial	Approximate Partial
15	Proper Exact	Approximate Exact
16	Approximate Exact	Approximate Complete
23	Proper	Enveloping
23	Proper Complete	Enveloping Complete
23	Proper Partial	Enveloping Partial
23	Proper Exact	Enveloping Exact
24	Enveloping Exact	Enveloping Complete

provable in R_1. Therefore, having Analogies for sufficient Suppositions in R, can result in having Analogies for Results in the subject matter.

Whence, it is sufficient to base an Analysis of Analogy on only Suppositions and Terms in Suppositions as we have done.

Finally, we would like to point out that we have not introduced notions concerning mixed analogy types. In particular, an Analogy along the lines of mixed Proper, Approximate and Enveloping Analogies may be distinguished. That is, we may have s_1, s_2, s_3, such that

$$\mathrm{Subs}(s_1/g(t),t) = f(s_1)$$

$$PS_1 \models \mathrm{Subs}(s_1/g(t),t) \rightarrow f(s_2)$$

and

$$PS_1 \models f(s_3) \rightarrow \mathrm{Subs}(s_1/g(t),t).$$

But we really need not concern ourselves with mixed strength analogies like this. Instead, we can break such an analogy down into three separate and distinct analogies: one Proper, one Approximate and one Enveloping. These Analogies would contain in R_1, only s_1, only s_2 and only s_3, respectively.

This completes our development and analysis of the notion of Analogy. It remains to discuss once again the use of Analogies in classifying subject matter content.

As mentioned earlier, Analogies can be employed profitably during instruction in one way by taking advantage of the learner's preexisting knowledge. This can be done by providing Analogies for new subject matter in terms of subject matter with which the learner is already familiar. For example, suppose a learner is to be taught a subject matter with Representation R. Suppose also the learner is familiar with another subject matter, a Representation of which, R_1, with functions f and g can serve as, say, an Approximate Partial Analogy. Ostensibly, it should be possible easily to acquaint the learner with relevant portions of R_1 and with the associations provided by f and g. Relevant instructional information can be explained in terms of R_1 and then carried back to R via f and g (i.e., their inverses). We expect that such ploys facilitate the learning process.

Another, already mentioned, use of Analogies in such an environment is to encourage hypothesis formulation. We expect that the learner's preexisting knowledge in the subject matter related to R_1 will lead to insightful formulation of principles. The learner can then transliterate these principles, via f and g, into R. The instructional agent can then, in concert with the learner, investigate the validity of these principles for R, or R's subject matter. Those principles that carry back to R provide examples of self-discovery on the learner's part. Those that do not carry back can be used by the instructional agent to point out salient features concerning R that invalidate the principles. This also is a positive aspect of the use of Analogy since it would be expected to result in an increased understanding of the principles underlying R. That is, the learner develops a keener appreciation of the divergence between R's subject matter and R_1's subject matter.

Finally, instruction by example or the use of examples to facilitate instruction is widely recognized and employed as a useful instructional tactic. Formally, an example is an Analogy. In other words, to structure an example is to establish an Analogy. Thus, we see that Analogies are recognized, though perhaps only implicitly recognized, as being useful during instruction.

This discussion leads to several interesting questions concerning the instructional use of Analogies. Just how useful are Analogies? When should they be employed and when not? Can we really find Analogies for different subject matters? How many Analogies might we be expected to find for various subject matters? Can we meaningfully classify subject matter content on the basis of potential or existing Analogies? What would be the potential use of such classifications?

At the outset we stress that many of these and related questions are not answerable at this time. Many of the questions would seemingly be answerable only on the basis of future empirical research and data. Thus, we recognize that our analysis of Analogy, in all probability, raises more questions than it answers, or even can be answered at present. However, we can shed some light on some of these problems.

Consider the photography example developed in Chapter I and its Representation. A full-blown Representation of photography can be expected to be based on numerous Suppositions, contain many Primary Terms, and in general, can be expected to be very complex. Indeed, the subject matter of photography is so domain-specific that we do not expect to find any Exact or Complete Analogies for a photography Representation. The only Analogy we expect to find in such a case is the intended interpretation of the subject matter: in this case in the world of cameras, light, film, etc. Thus, complex, interrelated subject matters are not expected to have

Exact or Complete Analogies except for their intended interpretations. In such cases, the instructional use of Analogy is probably restricted to the use of examples, or Analogies as examples.

On the other hand, different Partial Analogies for different subareas of photography are possible and potentially useful. A student's knowledge of optics, for example, might be capitalized upon in discussing lenses. A subject matter such as Huygens' theory of optics might serve as Partial Analogy for a part of the photography Representation. As another example, portions of Chemistry may be invoked as a Partial Analogy for some part of photography dealing with emulsions. Whence, we see that Partial Analogies for portions of complex subject matters are entirely feasible.

What form, Proper, Approximate, or Enveloping, might Partial Analogies take? We do not expect to find Proper Analogies for domain-specific, complex subject matters. There is simply no reason to expect a group of highly complex Suppositions to carry over, in tact, from one realm to another. To find such strong subject matter correlations strongly suggests that the subject matters are closely related. At least they will be relatable on the basis of the Analogy that is established. Such a result casts doubt on the assumption that the subject matters are domain-specific.

What about Enveloping Analogies? Surely Huygens' theory of light Analogically Envelops basic optics needed for photography. But is it necessary to establish such a strong Analogy? It seems more reasonable to be more modest and develop only an Approximate Analogy that will serve the instructional purposes. From the instructional agent's developmental point of view, Approximate Analogies will require less effort than will

Enveloping Analogies. From the instructional point of view, less learning, work, effort, etc., are required on the learner's part in comprehending Approximate rather than Enveloping Analogies. An Enveloping Analogy may even be so complex, as compared to the subject matter proper, that a tangential, and most undesirable, instructional effort may be required to make the learner comprehend the Analogical subject matter. Thus, it seems that, in general, Approximate Analogies will be more desirable than Enveloping Analogies.

The foregoing discussion is intended to suggest that Approximate Partial Analogies are the most useful type of Analogy for instructional purposes. However, an _a priori_ conjecture of this form must await _a posteriori_ evidence for confirmation. We can assert that, in most cases, instructional examples can be seen to be Approximate Partial Analogies. Thus, Approximate Partial Analogies are widely employed for instructional purposes. In other words, as now employed, instructionally, the preferred form of Analogy, is the Approximate Partial Analogy. This fact provides some evidence for our conjecture.

Classifying subject matter content on the basis of Analogical relationships, twelve of which have been identified above, is of limited value at this point in time. The reason is simply that we do not have a class of Axiomatically Structured Representations to work with. If we had such a class available, then presumably such a classification would be both possible and useful. Consider, for example, a situation such as the following. An instructional agent has "decided" to provide instruction in some subarea of some subject matter for a particular student. Perhaps this subarea is delineated through a Relatedness measure (defined earlier) on its components. Suppose also that the instructional agent has available a catalog of subject matters with which the learner is familiar, perhaps even with a familiarity measure for each one. Suppose also that the instructional agent has available a large collection of Representations of various subject matters. Conceivably, such a collection might be possible on a large computer network.

A natural question for the instructional agent to "ask" is: Do I have available an Analogy for this subject subarea, such that the learner is familiar, at some threshold level, with the Analogical subject matter? An affirmative response means that an Analogy is forthcoming that the instructional agent might utilize to capitalize on the learner's preexisting knowledge. At this point, the instructional agent must evaluate the potential usefulness and the suitability of the Analogy. As a result the Analogy may be adopted and brought into use minimizing the work required by the instructional agent and facilitating the learning process.

Thus, it is conceivable that classifying subject matter on the basis of Analogical Relationships is most useful. Basic relationships, once established, can be extended through the use of reflexive and transitivity properties discussed earlier (see Tables 1 and 2, respectively). We recognize that the most useful application of such a classification scheme may be in such large network environments. That is, as long as instructional efforts remain fragmented and duplicity is the order of the day, sophisticated techniques such as Analogical classifications may find limited applications, and may have limited impact.

However, even in an unlinked educational environment such as we find today, Analogical relationships are important. We can at least make more explicit use of Analogies during instruction. Analogical reasoning seems to be a preferred method of thinking, learning, and of organizing information. This last, and final point, will be amplified briefly.

Consider Euclidean geometry for a moment. As is well known, we can provide a parsimonious set of Primary Concepts and Suppositions for this subject matter. These axioms are the basis for the proof of all theorems in geometry. But notice that as soon as we are faced with a problem requiring a proof, what do we do? We pick up a pencil and sketch out the situation. We begin marking the sketch as we find equal lines, equal angles, congruent shapes, similar shapes, etc. In fact, we solve the problem (or prove the Result) in terms of the diagram. Our next step is to translate this "informal" reasoning process back into formal geometric terms--i.e., Primary Terms, Suppositions, Definitions, Results, etc. As a result, we construct a formal solution to the problem.

But notice that the diagram becomes at least an Approximate Partial Analogy. Perhaps the diagram is even either a Proper or Enveloping Partial Analogy. In any case, the essential point is that the insightful reasoning that leads to a successful solution to the problem is performed in the Analogical domain. Once the solution is perceived Analogically, it is a simple matter to translate the reasoning steps back into the subject matter proper. As a result a formal solution (or proof) is constructed.

Thus, we believe that Analogical reasoning is of paramount importance. Our Analogies provide us with fresh insight into old subject matters. Einstein's creative work might be seen to be an example of this process. Analogies also provide useful entries into new and unexplored subject areas. Indeed, we contend that Analogical relationships should be stressed and fully exploited instructionally since in Analogies we find the basis of our reasoning process. Analogical classificatory schemes are an essential ingredient in an instructional environment that maximizes the utilization of Analogies.

The above discussion places great emphasis on the importance of Analogies to our cognitive processes. In fact, we have asserted that Analogical reasoning is of primary importance in developing an adequate characterization or analysis of most subject areas. We will conclude our remarks on this subject by suggesting some evidence for this point of view. Artificial Intelligence research has focused, in part, on developing techniques that mimic human intelligent reasoning. A close look at systems intended to perform human-like reasoning shows that, almost without exception, the use of Analogy is ignored. Most Artificial Intelligent work on reasoning is based on an axiomatic approach. Axioms, etc., are provided and the system begins to develop hypotheses, check theorems, prove theorems, etc., in a strictly axiomatic, formal manner. Analogies simply play no part in these systems.

It is fairly well recognized that these research efforts have met with rather limited success. Human-like reasoning has not been achieved. We believe it will not be achieved as long as Analogical reasoning is ignored. That is, the failure of Artificial Intelligence efforts to achieve the goal of human-like reasoning, we suggest, may be due to the strict formalist approach that is usually taken, and that ignores the role of Analogy. This failure can be seen to support our contention that Analogical reasoning is a crucial ingredient in human reasoning: to ignore Analogy condemns any attempt to mimic human reasoning. This being the case, the instructional process should make explicit and exact use of Analogies whenever appropriate. An Analogical classificatory scheme for subject matter can play an important role in facilitating the use of Analogies in instruction.